J Syst Sci Complex (2012) 25: 594–616

A MAPLE PACKAGE OF AUTOMATED DERIVATION OF HOMOTOPY ANALYSIS SOLUTION FOR PERIODIC NONLINEAR OSCILLATIONS*

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DOI: 10.1007/s11424-012-0043-3
Received: 3 March 2010 / Revised: 25 September 2010
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Abstract Based on the homotopy analysis method, a general analytic technique for strongly nonlinear problems, a Maple package of automated derivation (ADHO) for periodic nonlinear oscillation systems is presented. This Maple package is valid for periodic oscillation systems in rather general, and can automatically deliver the accurate approximations of the frequency ω and the mean of motion δ of a nonlinear periodic oscillator. Based on the homotopy analysis method which is valid even for highly nonlinear problems, this Maple package can give accurate approximate expressions even for nonlinear oscillation systems with strong nonlinearity. Besides, the package is user-friendly: One just needs to input a governing equation and initial conditions, and then gets satisfied analytic approximations in few seconds. Several different types of examples are given in this paper to illustrate the validity of this Maple package. Such kind of package provides us a helpful and easy-to-use tool in science and engineering to analyze periodic nonlinear oscillations. And it is free to download the electronic version of this Maple package from the address "http://numericaltank.sjtu.edu.cn/sjliao.htm" once the paper is published publicly.

Key words Automated derivation, homotopy analysis method, homotopy Padé technique, nonlinear oscillation, Wu's elimination method.

1 Introduction

Nonlinear oscillations governed by nonlinear differential equations have been a topic to intensive research for many years. Considerable interests have been focused on the study of analytic solutions for nonlinear differential equations, for they may give more insights into internal aspects of nonlinear problems. However, there are very few nonlinear differential equations whose exact solutions have been discovered. Hence, it is often necessary to construct approximate solutions of nonlinear differential equations by means of analytical techniques.

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*This research is supported by the National Science Foundation of China under Grant No. 11071274.
◇ *This paper was recommended for publication by Editor Xiao-Shan GAO.*

Foremost among the analytic techniques are perturbation methods in terms of a small/large parameter[1−4]. Because many nonlinear problems do not contain such a small/large parameter (perturbation quantity), some non-perturbation methods, such as the artificial small parameter method, the δ-expansion method and adomian decomposition methods, were developed[5−7]. However, both of these perturbation and non-perturbation methods cannot provide a simple way to ensure the convergence of analytic approximations.

The essence of perturbation methods is to transfer a nonlinear problem by means of an infinite number of linear sub-problems. Unfortunately, restricted by the small/large physical parameters, the perturbation techniques cannot provide any freedom to choose the type of the linear sub-problems. For example, writing

$$u(t) = u_0(t) + \varepsilon\, u_1(t) + \varepsilon^2\, u_2(t) + \cdots$$

and using the perturbation method to solve the following periodic nonlinear oscillator:

$$u''(t) + \lambda u(t) = \varepsilon\, u^3(t), \quad u(0) = 1, u'(0) = 0,$$

we have a sequence of linear equations

$$\begin{aligned} &u_0''(t) + \lambda\, u_0(t) = 0, \quad u_0(0) = 1, u_0'(0) = 0, \\ &u_1''(t) + \lambda\, u_1(t) = -\varepsilon\, u_0^3(t), \quad u_1(0) = 0, u_1'(0) = 0, \end{aligned}$$

and so on. Unfortunately, when $\lambda < 0$, the equation

$$u_m''(t) + \lambda u_m(t) = 0$$

has a general solution

$$u_m(t) = c_1 \exp(-\sqrt{|\lambda|}t) + c_2 \exp(+\sqrt{|\lambda|}t),$$

which is not periodic at all! For details, please refer to Liao and Tan[8]. This simple example illustrates that, using perturbation techniques, we have nearly no freedom to choose the type of the sub-problems. And this often leads to great difficulties for many nonlinear oscillation problems.

In 1992, Liao[9] employed the basic ideas of the homotopy in topology to propose a general analytic method for highly nonlinear problems, namely, the homotopy analysis method (HAM). Unlike perturbation methods, the HAM does not need any small/large physical parameters. Besides, different from all other analytic techniques, the HAM provides us with a simple way to guarantee the convergence of approximate series solutions[8,10]. Furthermore, the HAM provides us great freedom to choose different types of base functions to approximate solutions of a given nonlinear equation, and thus we have great freedom to choose the types of the linear sub-problems. The above-mentioned freedom of the HAM implies that it is possible to develop some general symbolic computation codes for some sorts of nonlinear problems (we will show this point in this paper). The HAM has been successfully applied to solve many nonlinear problems in science and engineering[8−23]. Especially, some new solutions have been found, which have been neglected by other analytic techniques and even by numerical techniques, as shown by Liao[14]. All of these show the validity of the HAM. Thus, the HAM can overcome the limitations and restrictions of perturbation and traditional non-perturbation techniques, and is indeed valid for strongly nonlinear problems.

In science and engineering, one often needs to solve a periodic nonlinear oscillation quickly. Certainly, this can be done by several numerical codes. However, numerical results are discrete,

and it is not easy to have some deep physical insights from this kind of discrete numerical results. On the other sides, due to the above-mentioned restrictions of perturbation methods, there does not exist a symbolic computation code for general periodic oscillation systems with strong nonlinearity.

In this paper, we apply the HAM to give a symbolic computation code for general periodic oscillations with strong nonlinearity. For this purpose, we consider here free oscillations of a conservative system governed by

$$\ddot{U}(t) = f[U(t), \dot{U}(t), \ddot{U}(t)], \tag{1}$$

where t denotes the time, the dot denotes derivative with respect to t, and $f[U(t), \dot{U}(t), \ddot{U}(t)]$ is an algebraic and/or fractional function of $U(t), \dot{U}(t)$ and $\ddot{U}(t)$. It should be emphasized that Equation (1) is rather general and can be used to describe many problems in science and engineering. Especially, Equation (1) is unnecessary to contain any small/large physical parameters.

Physically speaking, free oscillation of a conservative system is a periodic motion. Let ω and a denote the frequency and amplitude of oscillation, respectively. Define the mean of motion

$$\delta = \frac{1}{T}\int_0^T U(t)dt, \tag{2}$$

where $T = 2\pi/\omega$ is the period of oscillation. For conservative system with odd nonlinearity, the mean of motion δ is equal to zero. Physically, the frequency ω can be regarded as a time scale. For a linear system, the frequency is independent of the amplitude. However, for a nonlinear system, it is very important to know the relationship between the frequency and the amplitude. In a nonlinear conservative system, the amplitude a is physically determined by initial conditions and is related to the total kinetic energy. Without the loss of any generality, we may consider free oscillations with amplitude a under the initial conditions

$$\dot{U}(0) = 0,\ U(0) = a + \delta. \tag{3}$$

By introducing the transformation $\tau = \omega\, t,\ U(t) = \delta + u(\tau)$. Equations (1) and (3) become

$$\omega^2\, \ddot{u}(\tau) = f[\delta + u(\tau), \omega\, \dot{u}(\tau), \omega^2\, \ddot{u}(\tau)] \tag{4}$$

and

$$u(0) = a, \quad \dot{u}(0) = 0. \tag{5}$$

We have developed a Maple package ADHO (Automated Derivation of the Homotopy analysis solutions for nonlinear Oscillation), which can entirely automatically derive analytic approximate solutions for equations in the form of (4) and (5), meanwhile, ADHO also delivers analytic approximate expression or high-precision approximation of ω and δ. The effectiveness of our package is demonstrated by different kinds of examples. It can be seen that the ADHO is an effective and also efficient package to derive analytic approximate solutions for nonlinear oscillations.

The paper is organized as follows. In Section 2, we describe the basic idea of the homotopy analysis method to construct analytic approximate solutions for free oscillations of conservative systems. In Section 3, a Maple package ADHO is presented. In Section 4, several different kinds of examples are given to demonstrate the effectiveness of our package. Finally, some discussions and conclusions are given.

2 The Homotopy Analysis Method

As HAM has been widely used to solve nonlinear problems in science and engineering, the rules and procedures of the HAM is expounded by Liao in [9] and [10]. In this section, free oscillations of conservative systems (4) and (5) are used as examples to show the basic ideas and procedures of the HAM.

2.1 Rule of Solution Expression

When the HAM is applied to solve a nonlinear differential system, first, a set of base functions are selected to represent the required solution. This is the so-called "Rule of solution expression". As free oscillations have been widely studied and so many papers have been reported to discuss solutions for different oscillation equations. Liao has done excellent work on this issue and obtained some very good results. Follow all these work, we choose base functions for Equations (4) and (5) with the form

$$\{\cos(m\,\tau)\mid m=1,\,2,\,\cdots\}. \tag{6}$$

2.2 Choosing Initial Guess and Auxiliary Linear Operator

It is under the rule of solution expression that the initial approximation, the auxiliary linear operator are selected. For nonlinear oscillations, the initial guess $u_0(\tau)$ and the auxiliary linear operator L are usually chosen as

$$u_0(\tau)=a\,\cos(\tau) \tag{7}$$

and

$$L[\Phi(\tau,q)]={\omega_0}^2\left(\frac{\partial^2\Phi(\tau,q)}{\partial\tau^2}+\Phi(\tau,q)\right) \tag{8}$$

with the property

$$L[C_1\sin(\tau)+C_2\cos(\tau)]=0, \tag{9}$$

where ω_0 is the initial guess of the frequency ω. $\Phi(\tau,q)$ is a function of τ and q, C_1, C_2 are integral constants.

2.3 The Zero-Order Deformation Equation

For Equation (4), the nonlinear operator is written as

$$\begin{aligned}N[\Phi(\tau,q),\Omega(q),\Delta(q)]&=\Omega(q)^2\,\frac{\partial^2\Phi(\tau,q)}{\partial\tau^2}\\&\quad-f\left(\Delta(q)+\Phi(\tau,q),\Omega(q)\,\frac{\partial\Phi(\tau,q)}{\partial\tau},\Omega(q)^2\,\frac{\partial^2\Phi(\tau,q)}{\partial\tau^2}\right),\end{aligned} \tag{10}$$

where $\Omega(q)$ and $\Delta(q)$ are functions of the embedding parameter $q\in[0,1]$, corresponding to the frequency ω and the mean of motion δ, respectively. It is to be stressed that if function f in (10) just contains odd nonlinearity, $\Delta(q)$ in (10) is equal to zero.

The homotopy analysis method is based on such continuous variations $\Phi(\tau,q)$, $\Omega(q)$ and $\Delta(q)$ that, as the embedding parameter q varies from 0 to 1, $\Phi(\tau,q)$ varies from the initial guess $u_0(\tau)$ to the exact solution $u(\tau)$, so does $\Omega(q)$ from the initial guess ω_0 to the exact frequency ω, and $\Delta(q)$ from the initial guess δ_0 to the exact mean of motion δ, respectively.

By introducing $q \in [0,1]$ as an embedding parameter, we construct a homotopy with the form

$$\begin{aligned}&\mathcal{H}(\Phi(\tau,q);\Omega(q),\Delta(q),q,h,H(\tau))\\&= (1-q)\{L[\Phi(\tau,q)-u_0(\tau)]\} - q\,h\,H(\tau)\,N[\Phi(\tau,q),\Omega(q),\Delta(q)],\end{aligned} \tag{11}$$

in which $h \neq 0$ is an auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function. They play important roles in the homotopy analysis method.

Enforcing the homotopy (11) to be zero, we obtain the so-called zero-order deformation equation

$$(1-q)\{L[\Phi(\tau,q)-u_0(\tau)]\} = q\,h\,H(\tau)\,N[\Phi(\tau,q),\Omega(q),\Delta(q)], \tag{12}$$

subject to the initial conditions

$$\Phi(0,q) = a, \quad \left.\frac{\partial\Phi(\tau,q)}{\partial\tau}\right|_{\tau=0} = 0. \tag{13}$$

When $q=0$, it is easy using (7) and (12) to show that

$$\Phi(\tau,0) = u_0(\tau), \quad \Omega(0) = \omega_0, \quad \Delta(0) = \delta_0. \tag{14}$$

When $q=1$, since $h \neq 0$ and $H(r,t) \neq 0$, the zero-order deformation equation (12) is equivalent to

$$N[\Phi(\tau,1),\Omega(1),\Delta(1)] = 0, \tag{15}$$

which is exactly the same as the original equation (4), provided

$$\Phi(\tau,1) = u(\tau), \quad \Omega(1) = \omega, \quad \Delta(1) = \Delta. \tag{16}$$

Thus, according to (14) and (16), as the embedding parameter q increases from 0 to 1, $\Phi(\tau,q)$ varies continuously from the initial guess $u_0(\tau)$ to the exact solution $u(\tau)$, so does $\Omega(q)$ from the initial guess ω_0 to the exact frequency ω, and $\Delta(q)$ from the initial guess δ_0 to the exact mean of motion δ.

Due to Taylor's theorem and (14), we expand $\Phi(\tau,q)$, $\Omega(q)$ and $\Delta(q)$ in the power series of q as follows:

$$\Phi(\tau,q) = u_0(\tau) + \sum_{n=1}^{\infty} u_n(\tau)\,q^n, \quad \Omega(q) = \omega_0 + \sum_{n=1}^{\infty}\omega_n\,q^n, \quad \Delta(q) = \delta_0 + \sum_{n=1}^{\infty}\delta_n\,q^n, \tag{17}$$

where

$$u_n(\tau) = \frac{1}{n!}\left.\frac{\partial^n\Phi(\tau,q)}{\partial q^n}\right|_{q=0}, \quad \omega_n = \frac{1}{n!}\left.\frac{\partial^n\Omega(q)}{\partial q^n}\right|_{q=0}, \quad \delta_n = \frac{1}{n!}\left.\frac{\partial^n\Delta(q)}{\partial q^n}\right|_{q=0}. \tag{18}$$

By properly choosing the auxiliary parameter h and the auxiliary function $H(\tau)$ so that the above series is convergent when q=1, we have due to (14) that

$$u(\tau) = u_0(\tau) + \sum_{n=1}^{\infty} u_n(\tau), \quad \omega = \omega_0 + \sum_{n=1}^{\infty}\omega_n, \quad \delta = \delta_0 + \sum_{n=1}^{\infty}\delta_n. \tag{19}$$

2.4 The Higher-Order Deformation Equation

For brevity, define the vectors

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}, \quad \vec{\omega}_n = \{\omega_0, \omega_1, \cdots, \omega_n\}$$

and

$$\vec{\delta}_n = \{\delta_0, \delta_1, \cdots, \delta_n\}.$$

Differentiating the zero-order deformation equation (12) n times with respect to q and then dividing them by $n!$ and finally setting $q = 0$, we have the so-called nth-order deformation equation

$$L[u_n(\tau) - \chi_n u_{n-1}(\tau)] = h\, H(\tau)\, R_n(\vec{u}_{n-1}, \vec{\omega}_{n-1}, \vec{\delta}_{n-1}), \tag{20}$$

subject to the initial conditions

$$u_n(0) = \dot{u}_n(0) = 0, \tag{21}$$

where

$$R_n(\vec{u}_{n-1}, \vec{\omega}_{n-1}, \vec{\delta}_{n-1}) = \frac{1}{(n-1)!} \frac{\partial^{n-1} N[\Phi(\tau, q), \Omega(q), \Delta(q)]}{\partial q^{n-1}}\bigg|_{q=0} \tag{22}$$

and

$$\chi_n = \begin{cases} 0, & n \le 1, \\ 1, & n > 1. \end{cases} \tag{23}$$

The nth-order deformation equation (20) is a linear equation for $u_n(\tau)$. However, apart from $u_n(\tau)$, there are two unknowns ω_{n-1} and δ_{n-1} contained in (20). So the problem is not closed, and two algebraic equations are needed to determine ω_{n-1} and δ_{n-1}.

Follow the rule of solution expression, the auxiliary function $H(\tau)$ might appear as

$$H(\tau) = \cos(2\, k\, \tau),$$

where k is an integer. For simplicity, we choose $H(\tau) = 1$. Under this assumption, according to the properties of trigonometric functions, the right-hand side of the nth-order deformation equation (20) can be expressed by

$$b_{n,0} + \sum_{n=1}^{\Phi(n)} b_{n,k} \cos(k\tau),$$

where the integer $\phi(n)$ is dependent on n and the form of the original equation (4).

According to the rule of solution expression, to avoid the so called secular terms, $b_{n,0}$ and $b_{n,1}$ must be vanished, from which two algebraic equations are established for ω_{n-1} and δ_{n-1}. Thus the problem is closed, and we can compute the components $u_n(\tau)$, ω_{n-1} and δ_{n-1}, successively. At the Nth-order approximation, we have

$$\tilde{u}(\tau) \approx \sum_{n=0}^{N} u_n(\tau), \quad \tilde{\omega} \approx \sum_{n=0}^{N} \omega_n, \quad \tilde{\delta} \approx \sum_{n=0}^{N} \delta_n. \tag{24}$$

We should emphasize, from above steps the obtained component $u_n(\tau)$ must be in the form

$$u_n(\tau) = C_1 \sin(\tau) + C_2 \cos(\tau) + \sum_{k=2}^{\phi(m)} b'_{m,k} \cos(k\tau),$$

where C_1 and C_2 are integral constants. To obey the rule of solution expression, C_1 must be zero. To ensure that the amplitude of oscillation equals to a, we use

$$u_n(0) - u_n(\pi) = 0, \quad n = 1, 2, \cdots,$$

which determines the coefficient C_2.

It is to be stressed that if f in (4) just contain odd nonlinearities, δ is equal to zero in the above. In this way, the $b_{m,0}$ mentioned above naturally equate zero, and we use the algebraic equation $b_{m,1} = 0$ to determine ω_{n-1}.

2.5 Determine the Auxiliary Parameter h

Follow the above steps, we gain a family of solution series in the auxiliary parameter h. How does one then to choose the value of h to ensure that the solution series converges fast enough in a large enough region? We determine h by computing the residual error of the Equation (4) as follows:

1) Calculate the residual error of the Equation (4)

$$\varepsilon(h,\tau) = N[\tilde{u}(\tau), \tilde{\omega}, \tilde{\delta}].$$

2) As free oscillation of a conservative system is a periodic motion. We further to calculate the integral

$$\Lambda(h) = \int_0^{2\pi} \varepsilon(h)^2 d\tau$$

to eliminate τ.

3) The obtained result $\Lambda(h)$ is a single variable continuous function with respect to h, plot the graph of $\Lambda(h)$ and find the minimal point on the curve. Thus the best value of h is determined.

2.6 The Homotopy-Padé Technique

Padé approximant expands a function as a ratio of two power series. If the rational function is

$$R(x) = \frac{\sum\limits_{k=0}^{M} a_k\, x^k}{1 + \sum\limits_{k=1}^{N} b_k\, x^k}, \tag{25}$$

then $R(x)$ is said to be a Padé approximant to the series

$$f(x) = \sum_{k=0}^{\infty} c_k\, x^k, \tag{26}$$

if

$$\begin{aligned} &R(0) = f(0), \\ &\frac{d^k}{dx^k} R(x)\bigg|_{x=0} = \frac{d^k}{dx^k} f(x)\bigg|_{x=0}, \quad k = 1, 2, \cdots, M+N. \end{aligned} \tag{27}$$

Equation (27) provides $M+N+1$ equations for the unknowns a_0, a_1, $\cdots$, a_M and b_1, b_2, $\cdots$, b_N. The Padé technique can be employed within the frame of the homotopy analysis method. Such as for the problem we considered, it is very important to ensure that the series (17) is convergent at $q = 1$, for only in this way the solution series (19) can be obtained. We employ the traditional Padé technique to the first series in (17) about the embedding parameter q to

gain the $[m, n]$ Padé approximant

$$\frac{\sum\limits_{k=0}^{m} W_k(\tau) q^k}{1 + \sum\limits_{k=1}^{n} W_{m+k}(\tau) q^k},$$

where $W_k(\tau), k = 0, 1, \cdots, m+n$, are functions determined by the first several components $u_j(\tau),\ j = 0,\ 1,\ 2,\ \cdots, m+n$, following the formula (27). Then using (16), we set $q = 1$ to get the so-called $[m, n]$ homotopy-Padé approximant

$$\frac{\sum\limits_{k=0}^{m} W_k(\tau)}{1 + \sum\limits_{k=1}^{n} W_{m+k}(\tau)}.$$

It is found that the homotopy-Padé approximant often converges faster than the corresponding traditional $[m, n]$ Padé approximant. In many cases, the $[m, m]$ homotopy-Padé approximant does not depend upon the auxiliary parameter h. The homotopy-Padé technique can greatly enlarge the convergence region and rate of solution series.

3 The Package ADHO

Although the HAM mentioned in Section 2 is relatively simple in principle, it will be very complicated by hand. Fortunately, the HAM is an algorithmic method. In this paper, we present a Maple package ADHO, which can entirely automatically derive homotopy analysis solutions for nonlinear oscillation equations in the form of (4) and (5).

We should emphasize that f in Equation (4) is a general function with respect to $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$. Naturally, f can be different types of functions, such as f is a polynomial about $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$, or f is a rational function for $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$, or f is a piecewise function with respect to $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$, etc.. However, for the majority of free oscillations of a conservative system, the function f in Equation (4) is a polynomial or a rational function about $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$. For the current version of our package, ADHO just works for the case when f is a polynomial or a rational function about $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$.

In ADHO, the main interface is $main(eqn, ini_con, [h_order, sol_order])$, in which the eqn represents an oscillation equation to be solved, the ini_con represents initial conditions satisfied by the free oscillation. The other two parameters h_order (the default value is 8) and sol_order (the default value is 12) are optional, which are used to control the process flow of our program. In order to meet different needs, the computation in our program are divided into two parts, in which the first part is dependent on the auxiliary parameter h, and the second part is free of the auxiliary parameter h. As described in the following:

1) If there are unknown parameters contained in eqn or ini_con. Then we just perform the first part computation, namely, we calculate the first h_order components of solution series as approximations of the required solution. In this case, our program will output the $[h_order/2, h_order/2]$ homotopy-Padé approximate expression of the frequency ω as well as the mean of motion δ.

2) If there isn't any unknown parameters contained in eqn or ini_con, then we first get the first h_order components of solution series by the first part computation, and further to calculate the residual error of the input equation to determine the value of h. Once the value

of h is determined, we may get a more accurate approximate solution (i.e., continue calculate from $h_order+1$ to sol_order order) by the second part computation. In this case, our program will output different order approximations and the corresponding errors of the frequency ω, so does the mean of motion δ. Meanwhile, it also delivers the comparison graphs of different order approximations of the required solution.

We should mention that the output of our program is flexible and diverse. Apart from all the output results listed in the next section, if necessary, users can also get components of the required solution series. For in our program we store components of a solution series in a set, and specify the attribute of its name as export. In this way, when we completed solving an example. We can list the related components sets, all these sets are outlined as follows:

- Know_u denotes the components set of $u(\tau)$.
- Know_omega denotes the components set of the frequency ω.
- Know_delta denotes the components set of the mean of motion δ.
- Pade_omega denotes the homotopy-Padé approximations set of ω.
- Pade_delta denotes the homotopy-Padé approximations set of δ.

To show contents of the above sets is very simple, such as for the set Know_u, one just needs to input the set name Know_u after the prompt ">", that is, >Know_u; More details about how to load our package and how to use it is demonstrated in Appendix A. All the procedures contained in the package ADHO are outlined in Appendix B.

4 The Application of ADHO

In this section, some examples are given to show the effectiveness of our package. For the limitation of the length of paper, we just consider three different types of nonlinear oscillation systems. We have solved more than 20 oscillation equations by using the package ADHO, for the most majority of examples considered, the running time is no more than 500 seconds on a laptop.

4.1 Example 1

Consider a free oscillation with odd nonlinearity[10]

$$\ddot{U} + U + \varepsilon U^3 = 0, \tag{28}$$

subject to the initial condition

$$U(0) = a, \quad \dot{U}(0) = 0, \tag{29}$$

where $U = U(t)$, ε, a are unknown parameters.

This problem contains two physical parameters ε and a. If the input governing equation and the initial conditions contain any unknown physical parameters, our package automatically delivers the homotopy-Padé approximations for the frequency ω. For example, in case of both ε and a are not assigned, i.e., they are symbols, at the 5th-order of approximation (given by $h_order = 5$), the package outputs the $[1, 1]$ homotopy-Padé approximation:

$$\omega_{P_{1,1}} = \frac{(279\,\varepsilon^2\,a^4 + 768\,\varepsilon\,a^2 + 512)\,\sqrt{3\,\varepsilon\,a^2 + 4}}{2\,(285\,\varepsilon^2\,a^4 + 768\,\varepsilon\,a^2 + 512)},$$

and the [2, 2] homotopy-Padé approximation:

$$\begin{aligned}\omega_{P_{2,2}} = [&574118847\,\varepsilon^7 a^{14} + 5452491348\,\varepsilon^6 a^{12} + 22121717760\,\varepsilon^5 a^{10}\\ &+ 49711251456\,\varepsilon^4 a^8 + 66831974400\,\varepsilon^3 a^6 + 53758656512\,\varepsilon^2 a^4\\ &+ 23957864448\,\varepsilon a^2 + 4563402752\big]/\big[2\sqrt{3\,\varepsilon a^2 + 4}(195618321\,\varepsilon^6 a^{12}\\ &+ 1584935424\,\varepsilon^5 a^{10} + 5335944192\,\varepsilon^4 a^8 + 9556131840\,\varepsilon^3 a^6 + 9602662400\,\varepsilon^2 a^4\\ &+ 5133828096\,\varepsilon a^2 + 1140850688)\big].\end{aligned}$$

In case of $a = 1/2$, there is only one unknown physical parameter ε, our package automatically delivers (by $h_order = 5$) the [1,1] homotopy-Padé approximation:

$$\omega_{P_{1,1}} = \frac{(279\,\varepsilon^2 + 3072\,\varepsilon + 8192)\sqrt{3\,\varepsilon + 16}}{4\,(285\,\varepsilon^2 + 3072\,\varepsilon + 8192)},$$

and the [2,2] homotopy-Padé approximation:

$$\begin{aligned}\omega_{P_{2,2}} = [&574118847\,\varepsilon^7 + 21809965392\,\varepsilon^6 + 353947484160\,\varepsilon^5 + 3181520093184\,\varepsilon^4\\ &+ 17108985446400\,\varepsilon^3 + 55048864268288\,\varepsilon^2 + 98131412779008\,\varepsilon\\ &+ 74766790688768\big]/\big[4\sqrt{3\,\varepsilon + 16}(195618321\,\varepsilon^6 + 6339741696\,\varepsilon^5 + 85375107072\,\varepsilon^4\\ &+ 611592437760\,\varepsilon^3 + 2458281574400\,\varepsilon^2 + 5257039970304\,\varepsilon + 4672924418048)\big].\end{aligned}$$

Besides, our package automatically gives a figure to compare approximations of ω at different order. As shown in Figure 1, the above [1,1] and [2,2] homotop-Padé approximations of ω agree quite well even for $0 \le \varepsilon < +\infty$.

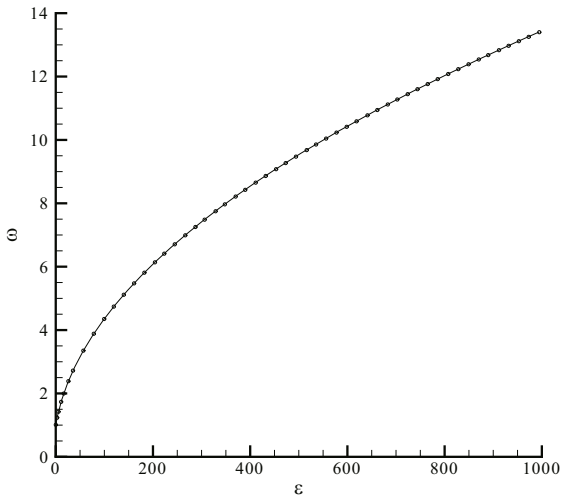

Figure 1 The homotopy-Padé approximant of the frequency ω of Equation (28) as a function of ε. Solid line: [1,1] homotopy-Padé approximant; symbols: [2,2] homotopy-Padé approximant

To further show the convergence of solution series, we consider the case of $\varepsilon = 2$ and $a = 1/2$. In this case, all physical parameters are known, and our package automatically delivers the homotopy series approximation as well as homotopy-Padé approximants of the

frequency ω. Meanwhile, our package also gives the residual error curves as well as a figure for the comparison of different-order approximations of $u(\tau)$. For example, using $h_order = 5$ and $sol_order = 7$, our package outputs the following figure and table within 6 seconds.

Figure 2 shows the square residual error of Equation (28) versus the auxiliary parameter h. From Figure 2, it is clear that $h = -1$ corresponds to the minimum square residual error. So, we choose $h = -1$ here. As shown in Figure 3, the 5th and 7th-order approximations of $u(\tau)$ given by $h = -1$ agree very well. Besides, the ith-order approximation of ω and its corresponding $[i, i]$ homotopy-Padé approximants are given in Tables 1 and 2, respectively. Obviously, the homotopy-Padé approximant of ω converges faster than the homotopy series approximation, as shown in Tables 1 and 2.

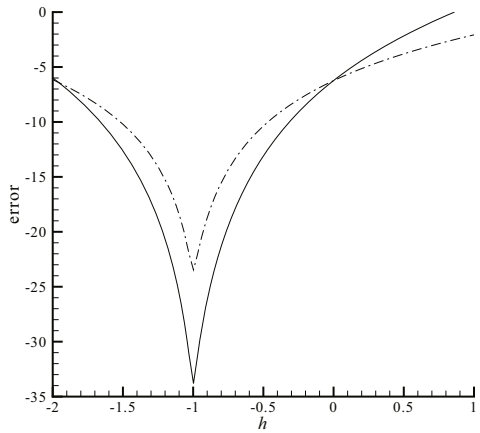

Figure 2 The residual errors of Equation(28) when $\varepsilon = 2$ and $a = 1/2$. Solid line: 3th-order HAM approximation; symbols: 5th-order HAM approximation

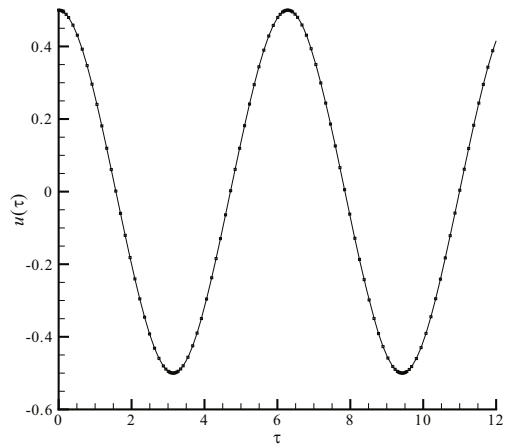

Figure 3 The HAM approximations of $u(\tau)$ for Equation(28) when $\varepsilon = 2$ and $a = 1/2$ by means of $h = -1$. Solid line: 5th-order HAM approximation; symbols: 7th-order HAM approximation

Table 1 The ith-order approximations of ω for Equation (28) when $\varepsilon = 2$ and $a = 1/2$ by means of $h = -1$

i	ω
1	0.7905
2	0.7846
3	0.7846
4	0.7845
5	0.7845
6	0.7845

Table 2 The $[i, i]$ homotopy-Padé approximations of ω for Equation (28) when $\varepsilon = 2$ and $a = 1/2$

i	ω
1	0.7845
2	0.7845
3	0.7845

4.2 Example 2

For a periodic oscillator with quadratic nonlinearity, our package automatically gives approximations of both the frequency ω and the mean of motion δ. For example, let us consider a free oscillation with quadratic nonlinearity[10]

$$\ddot{U}(t) + U(t) + \varepsilon U^2(t) = 0, \tag{30}$$

subject to the initial condition

$$U(0) = 1/2, \quad \dot{U}(0) = 0, \tag{31}$$

where ε is a physical parameter.

When ε is unknown, our package automatically delivers (by $h_order = 7$) within 92 seconds the $[1, 1]$ homotopy-Padé approximants:

$$\omega_{P_{[1,1]}} = \frac{\sqrt{2}(190\,\varepsilon^2 - 47\,\varepsilon^4 - 192)}{(4 - 2\,\varepsilon^2)^{(3/4)}\,(49\,\varepsilon^2 - 96)}, \quad \delta_{P_{[1,1]}} = \frac{-2 + \sqrt{4 - 2\,\varepsilon^2}}{4\,\varepsilon},$$

the $[2, 2]$ homotopy-Padé approximants:

$$\omega_{P_{[2,2]}} = \frac{\sqrt{2}(480727\,\varepsilon^8 - 3951086\,\varepsilon^6 + 12173696\,\varepsilon^4 - 16665088\,\varepsilon^2 + 8552448)}{(4 - 2\,\varepsilon^2)^{(3/4)}\,(4276224 - 503415\,\varepsilon^6 + 3079552\,\varepsilon^4 - 6283520\,\varepsilon^2)},$$

$$\delta_{P_{[2,2]}} = -[467\,\varepsilon^6 - 3014\,\varepsilon^4 + 465\,\sqrt{4 - 2\,\varepsilon^2}\,\varepsilon^4 + 6464\,\varepsilon^2 - 2080\,\sqrt{4 - 2\,\varepsilon^2}\,\varepsilon^2$$
$$- 4608 + 2304\,\sqrt{4 - 2\,\varepsilon^2}]/[2\,\sqrt{4 - 2\,\varepsilon^2}\,(465\,\varepsilon^4 - 2080\,\varepsilon^2 + 2304)\,\varepsilon],$$

and the $[3, 3]$ homotopy-Padé approximants:

$$\omega_{P_{[3,3]}} = \sqrt{2}(1130668627705\,\varepsilon^{12} - 14070595057184\,\varepsilon^{10} + 72900980650752\,\varepsilon^{8} - 201291747278848\,\varepsilon^{6} + 312414105305088\,\varepsilon^{4} - 258427607580672\,\varepsilon^{2} + 89013331427328)(4 - 2\,\varepsilon^{2})^{(1/4)} \big/ [2(1184646184839\,\varepsilon^{12} - 14617933686496\,\varepsilon^{10} + 75121335880960\,\varepsilon^{8} - 205795997351936\,\varepsilon^{6} + 316983474847744\,\varepsilon^{4} - 260282051985408\,\varepsilon^{2} + 89013331427328)],$$

$$\delta_{P_{[3,3]}} = (2658507\sqrt{4 - 2\,\varepsilon^{2}}\,\varepsilon^{8} - 5294322\,\varepsilon^{8} + 44773248\,\varepsilon^{6} - 22432000\sqrt{4 - 2\,\varepsilon^{2}}\,\varepsilon^{6} + 70893792\sqrt{4 - 2\,\varepsilon^{2}}\,\varepsilon^{4} - 141696832\,\varepsilon^{4} - 99464192\sqrt{4 - 2\,\varepsilon^{2}}\,\varepsilon^{2} + 198928384\,\varepsilon^{2} - 104546304 + 52273152\sqrt{4 - 2\,\varepsilon^{2}}) \big/ [4\,\varepsilon\,(2647161\,\varepsilon^{8} - 22386624\,\varepsilon^{6} + 70848416\,\varepsilon^{4} - 99464192\,\varepsilon^{2} + 52273152)].$$

It is found that even the $[1, 1]$ homotopy-Padé approximant of the frequency ω and the mean of motion δ are rather accurate, as shown in Figures 4 and 5.

Besides, in case of $\varepsilon = 1/2$, there is no any unknown physical parameters and our package automatically derives (by $h_order = 7$ and $sol_order = 9$) Figures 6, 7 and Tables 3 and 4 within 124 seconds. Figure 6 gives the square residual error versus h which indicates that the best value of h is about -0.49. Using this value, the 7th and 9th-order approximations of $u(\tau)$ agree very well, as shown in Figure 7. Table 3 indicates that the frequency ω as well as the mean of motion δ converge quickly by means of $h = -0.49$. And their homotopy-Padé approximants converge even faster, as shown in Table 4.

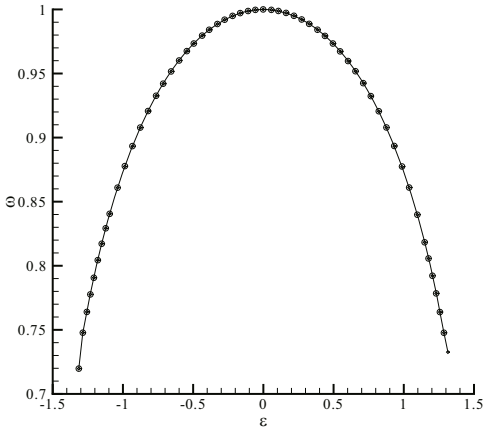

Figure 4 The homotopy-Padé approximants of ω for Equation (30) as a function of ε. Solid line: [1,1] homotopy-Padé approximant; Circles: [2,2] homotopy-Padé approximant; Diamonds: [3,3] homotopy-Padé approximant

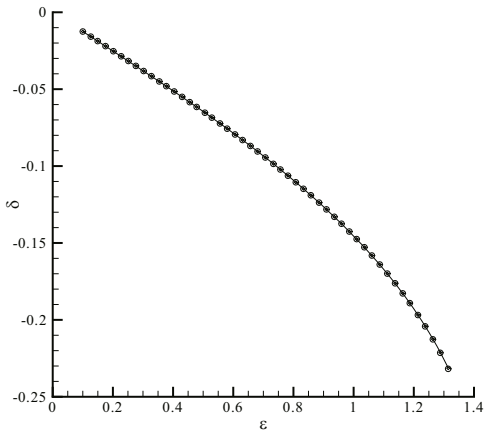

Figure 5 The homotopy-Padé approximants of δ for Equation (30) as a function of ε. Solid line: [1,1] homotopy-Padé approximant; Circles: [2,2] homotopy-Padé approximant; Diamonds: [3,3] homotopy-Padé approximant

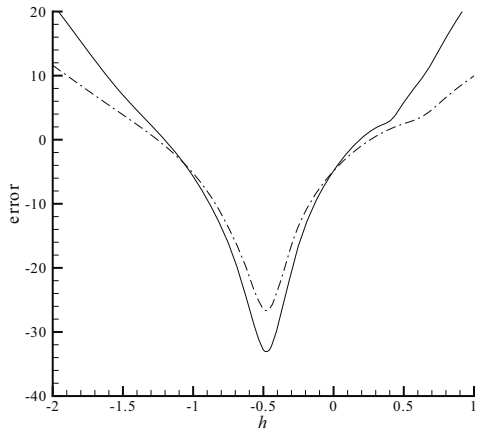

Figure 6 The residual errors curve of Equation (30) when $\varepsilon = 1/2$. Solid line: 5th-order approximation, dashed line: 7th-order approximation

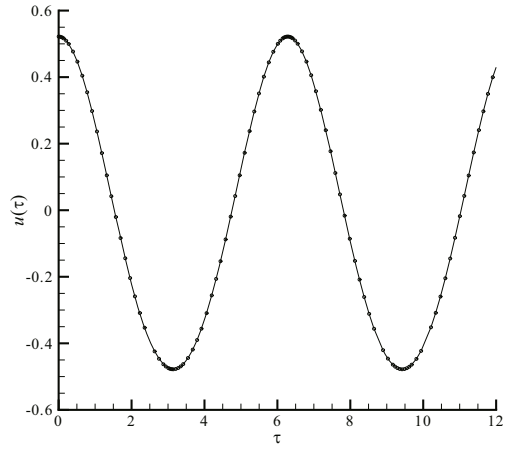

Figure 7 The approximation of $u(\tau)$ when $\varepsilon = 1/2$ by means of $h = -0.49$. Solid line: 7th-order HAM approximation, symbols: 9th-order HAM approximation

Table 3 The approximations of ω and δ for Equation (30) when $\varepsilon = 1/2$ by means of $h = -0.49$

i	ω_i	δ_i
1	0.9671	−0.0645
2	0.9729	−0.0646
3	0.9729	−0.0645
4	0.9728	−0.0645
5	0.9728	−0.0645
6	0.9728	−0.0645
7	0.9728	−0.0645
8	0.9728	−0.0645

Table 4 The homotopy-Padé approximations of ω and δ for Equation (30) when $\varepsilon = 1/2$

i	[Padé]ω_i	[Padé]δ_i
1	0.9729	−
3	0.9728	−0.0645
4	0.9728	−0.0645

4.3 Example 3

Our package is valid for fractional functions f of $U, \dot{U}$ and $\ddot{U}$. For example, let us consider

a free periodic oscillator governed by:

$$\ddot{U} = -\frac{\lambda U + \varepsilon_1 U\dot{U}^2 + 2\varepsilon_2 U^3\dot{U}^2 + \varepsilon_3 U^3 + \varepsilon_4 U^5}{1 + \varepsilon_1 U^2 + \varepsilon_2 U^4}, \tag{32}$$

subject to the initial condition

$$U(0) = a, \quad \dot{U}(0) = 0, \tag{33}$$

where λ is an integer which may take values -1, 0 or 1, a is amplitude, $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$ and $\varepsilon_4 > 0$ are physical parameters. Obviously, Equation (33) contains odd nonlinearity, and hence the mean of motion δ is zero.

In [24], its approximate solution were constructed by a series of skilled differential transformation. For this example, we solve it by our package in the following three cases.

4.3.1 All Physical Parameters Are Unknown

When setting *h_order* = 3, our program automatically delivers the [1, 1] homotopy-Padé approximate expression of ω within 30 seconds. However, the obtained approximate expression is so complicated that we omit it here for the limitation of the length of paper.

4.3.2 In Case of $\lambda = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$ and $a = 1$

In this case, there is only one unknown physical parameter ε_4. Using *h_order* = 5, our package takes 326.813 seconds to obtain the [1,1] homotopy-Padé approximant

$$\omega_{P_{1,1}} = \frac{\sqrt{210 + 75\,\varepsilon_4}\,(7276750\,{\varepsilon_4}^3 + 51116725\,{\varepsilon_4}^2 + 110797475\,\varepsilon_4 + 73257437)}{15\,(7276750\,{\varepsilon_4}^3 + 49994475\,{\varepsilon_4}^2 + 107990175\,\varepsilon_4 + 71501827)},$$

and the [2,2] homotopy-Padé approximant

$$\begin{aligned}
\omega_{P_{2,2}} = {} & (14 + 5\,\varepsilon_4)(3849275220595312500000\,{\varepsilon_4}^{10} + 8120226574090812500000\,{\varepsilon_4}^9 \\
& + 7535220781090250781250\,{\varepsilon_4}^8 + 4050788581240152557812 50\,{\varepsilon_4}^7 \\
& + 1398854338173871321406 25\,{\varepsilon_4}^6 + 3254329713150447039846875\,{\varepsilon_4}^5 \\
& + 5205601453787624730868125\,{\varepsilon_4}^4 + 5731361318662508515596875\,{\varepsilon_4}^3 \\
& + 4240192783183311382780900\,{\varepsilon_4}^2 + 1944636935808766840281615\,\varepsilon_4 \\
& + 4249793949218325772024 83)/[\sqrt{210 + 75\,\varepsilon_4}(3849275220595312500000\,{\varepsilon_4}^{10} \\
& + 8056573625355875000000\,{\varepsilon_4}^9 + 7431618357090107656250 0\,{\varepsilon_4}^8 \\
& + 3977695222032170589843750\,{\varepsilon_4}^7 + 1369472397725787948703125\,{\varepsilon_4}^6 \\
& + 3179806105582320673740625\,{\varepsilon_4}^5 + 5080642080241866604360625\,{\varepsilon_4}^4 \\
& + 5590479487994221881503625\,{\varepsilon_4}^3 + 4134973164483226215605000\,{\varepsilon_4}^2 \\
& + 1896525374159912031378565\,\varepsilon_4 + 4146647543462825698111 07)].
\end{aligned}$$

The above two approximations agree well for $0 < \varepsilon_4 < \infty$, as shown in Figure 8.

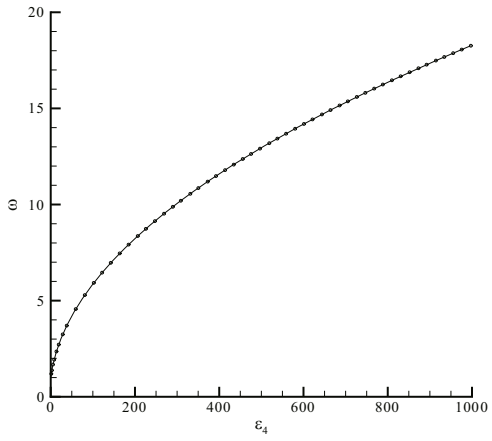

Figure 8 The homotopy-Padé approximants of Equation (32) in case of $\lambda = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $a = 1$. Line: [1,1] homotopy-Padé approximant; Symbols: [2,2] homotopy-Padé approximant

4.3.3 In Case of $\lambda = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $a = 1$

In this case, there is no unknown physical parameters and our program automatically derives within 754.672 seconds Figures 9, 10 and Tables 5 and 6.

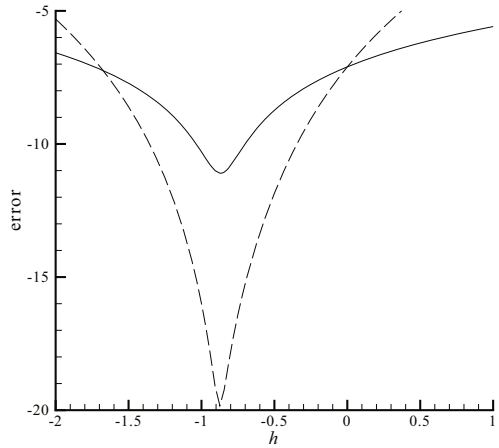

Figure 9 The residual errors of Equation (32) when $\lambda = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $a = 1$. Solid line: 1st-order HAM approximation; dashed line: 3rd-order HAM approximation

Table 5 The ith-order HAM approximation of ω for Equation (32) when $\lambda = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $a = 1$ by means of $h = -0.88$

i	ω
1	1.0334
2	1.0343
3	1.0349
4	1.0349

Table 6 The $[i, i]$ homotopy-Padé approximants of ω for Equation (32) when $\lambda = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $a = 1$

i	ω
1	1.0349
2	1.0349

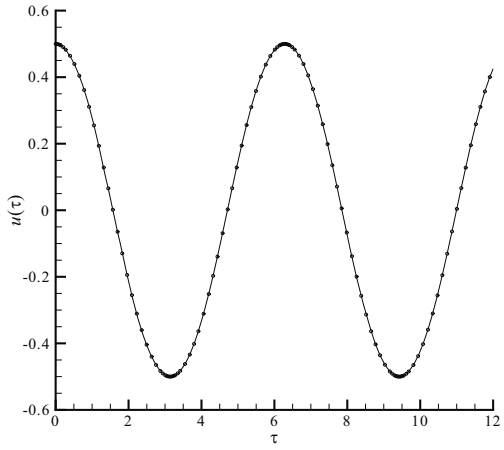

Figure 10 The HAM approximations of u for Equation(32) when $\lambda = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $a = 1$ by means of $h = -0.88$. Solid line: 3th-order HAM approximation; Symbols: 5th-order HAM approximation.

Figure 9 gives the residual errors of Equation (32) versus h, which suggests that the best value of the auxiliary parameter is about $h = -0.88$. Using $h = -0.88$, the HAM approximations converge quickly, as shown in Figure 10. Besides, the frequency converges also very quickly, as shown in Tables 5 and 6.

5 Discussions and Conclusions

In this paper, based on the homotopy analysis method and the homotopy-Padé technique,

a Maple package ADHO for periodic oscillation systems with strong nonlinearity is presented. The package ADHO can automatically derive accurate approximations of various types of nonlinear oscillation equations, as illustrated in this paper. In order to meet different needs, the output of the package is flexible and diverse. If there are unknown physical parameters in the input governing equations or initial conditions, the package ADHO automatically delivers the homotopy-Padé approximate expressions of the frequency (and the mean of motion in case of quadratic nonlinearity). If all physical parameters are known, the package ADHO firstly calculates the first *h_order* components of solution series with the unknown auxiliary parameter h, and then gives a figure for the residual error versus h, from which we can choose a good enough value of h. Once the value of h is determined and assigned, the package ADHO gives the higher-order approximations. Thereafter, the package ADHO automatically delivers the homotopy approximations as well as the homotopy-Padé approximations of the frequency ω (and the mean of motion δ in case of quadratic nonlinearity). Meanwhile, it also shows us curves of residual errors at different order of approximations, and a figure for the comparison of different-order approximations. Furthermore, one can get even higher-order approximations, if necessary. In order to overcome the rapid expansion of middle expressions the divide-and-conquer algorithm, freeze technique and others were employed frequently in the package. The package ADHO has been successfully used to solve more than 20 different types of nonlinear oscillation equations, and mostly one can get accurate enough approximations by a laptop within 500 seconds.

However, as mentioned in Section 3, the package ADHO currently can work only when f in Equation (4) is a polynomial or a rational function with respect to $u(\tau)$, $\dot{u}(\tau)$ and $\ddot{u}(\tau)$. In theory, f can be more complicated functions, such as piecewise function, or a function with absolute value function, or trigonometric functions, and so on. In future, we will further improve the package ADHO so that it can be applied to more general nonlinear oscillation equations.

Finally, it should be emphasized that our Maple package is valid for periodic oscillation systems in rather general, and can automatically deliver the accurate approximations of the frequency ω and the mean of motion δ of a periodic oscillator with strong nonlinearity. Besides, the package is user-friendly: one need just input a governing equation and initial conditions, and then get satisfied analytic approximations in few seconds. Such kind of package provides us a helpful and easy-to-use tool in science and engineering to analyze periodic nonlinear oscillations.

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Appendix A

As an example to show how to use our package ADHO, we consider a free oscillation just contain odd nonlineariry

$$\ddot{U} = -U + 2U^3 \tag{34}$$

subject to the initial condition

$$U(0) = 1/2, \ \dot{U}(0) = 0. \tag{35}$$

To load the package ADHO, one can proceed as follows:

$> restart:$ initializing Maple.

$> currentdir(\text{“}D:/ect001a\text{”});$ setting current work directory.

Please note, the program and sample files should be located into current directory.

$> read$ ‘*oscillation.mpl*’: reading the program file into memory

$> with(ADHO);$ loading the package ADHO.

To solve the system (34) and (35), one can run the main procedure as follows:

$> main(\text{diff}(U(t), t\$2) = -U(t) + 2 * U(t)^3, [U(0) = 1/2, D(U)(0) = 0], 5, 8);$

The input equation and initial conditions are

$$\frac{\partial^2}{\partial t^2}U(t) + U(t) - 2U^3(t),$$

$$U(0) = 1/2,\ D(U)(0) = 0.$$

The new EQ and initial conditions in u with the form:

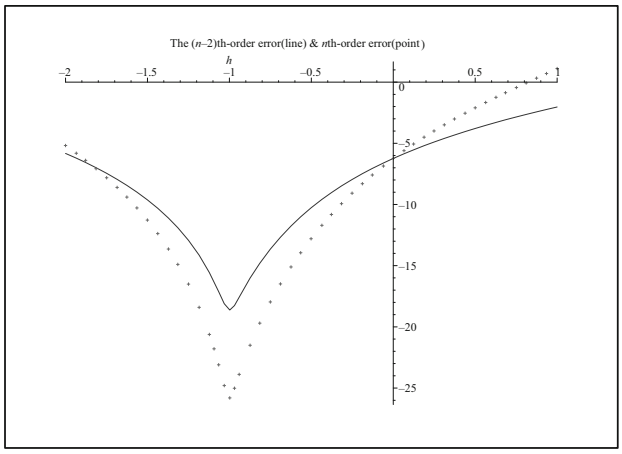
$$w^2\frac{\partial^2}{\partial \tau^2}u(\tau) + u(\tau) - 2u^3(\tau),$$

$$u(0) = 1/2,\ D(u)(0) = 0,$$

where

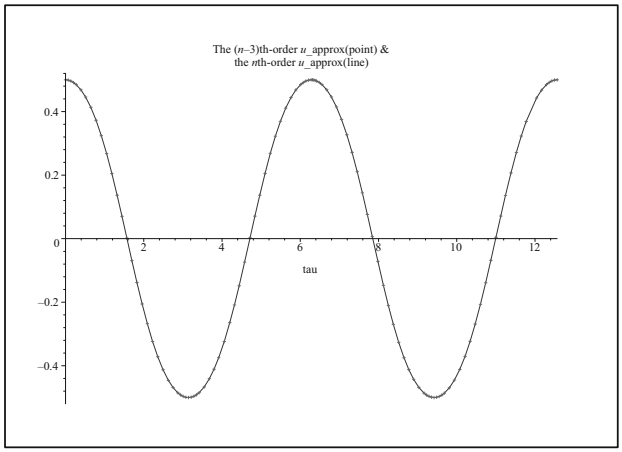
$$\tau = \omega * t, \quad U(t) = u(\tau).$$

It take about 3.797 seconds to complete the first part computation.



Where: $n = 5$.
Continue to adjust the domain for $h(Y/N)$?
$> n$
Please select a proper value for h!
> -1.0

Where: $n = 8$.



The approximations of ω and the corresponding errors:

i	ω_i	errors
1	0.7905	− − −
2	0.7846	0.0075
3	0.7846	0.0000
4	0.7845	0.0000
5	0.7845	0.0000
6	0.7845	0.0000

The approximants of $\omega_{Padé}$ and the corresponding errors:

i	$[Padé]\omega_i$	errors
1	0.7845	− − −
2	0.7845	0.0000
3	0.7845	0.0000

It takes about 0.890 seconds to complete the second part computation.
Continue to compute higher order approx(Y/N)?
$> n$
The total time is:10.828 seconds.

Appendix B

The package ADHO is comprised by 7 different procedures, among which are main, De_oper, Approx_nth, sol_analy, error_curve, conti_higher, proc_pade. We outline each of them as follows:

- main: This is the main procedure, its main task is to initialize relevant variables and to call other procedures.

- De_oper: The main task of this procedure is to change variables, and to define linear as well as nonlinear operators. Meanwhile, the input equation is decomposed to extract dependent and independent variables as well as unknown parameters.

- Approx_nth: In this procedure, follow the process and steps in Section 2, we compute the first *h_order* components(including $u_i(\tau)$, ω_{i-1}, δ_{i-1}, $i = 1, 2, \cdots, h_order$) of the solution series.

- sol_analy: In this procedure, we first normalize all solution expressions. If there are unknown parameters contained in *eqn* or *ini_con*, we call the function proc_pade to compute the homotopy-Padé approximations of the frequency ω as well as the mean of motion δ. If there isn't any unknown parameters contained in *eqn* or *ini_con*, we compute the residual error of the input equation, the obtained results is a function denoted by $\Delta(h, \tau)$. Further to eliminate τ by integrating Δ^2 with respect to τ from 0 to 2π, the obtained result is denoted as $\Lambda(h)$. By looking for the minimal value of $\Lambda(h)$ to determine the value of h.

- error_curve: The main task of this procedure is to adjust the domain of h, so that the minimal point of the curve $\Lambda(h)$ can be seen clearly.

- conti_higher: The main task of this procedure is to continue to get more accurate approximations of the required solutions. It is to be stressed that this part of the calculation is free of the auxiliary parameter h. Evidently, only when Equations (4) and (5) do not contain any parameters, the function conti_higher will be called.

- proc_pade: This procedure is a complete implementation of the homotopy-Padé technique.