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Parallel Computation of Reliable Chaotic Solutions of Saltzman's Equations by Means of the Clean Numerical Simulation

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Abstract

The method of the so-called "Clean Numerical Simulation" (CNS) is applied to gain reliable chaotic solutions of Saltzman's dynamic system, a simplified model for convection flows of fluid. Based on the high-order Taylor series method with data in multiple precision library and a validation check of global reliability of result, the CNS provides us a practical way to gain reliable, accurate enough solutions of chaotic dynamic systems in a finite but long enough time interval. The parallel computation is used to greatly increase the computational efficiency. The numerical noises of the CNS can be controlled to be so small that even the influence of the micro-level inherent uncertainty of initial conditions can be investigated in details. It is found that the micro-level inherent physical uncertainty (i.e. the unavoidable statistical fluctuation of temperature and velocity of fluid) of initial conditions of chaotic Saltzman's system transfers into macroscopic randomness. This suggests that chaos might be a bridge between micro-level inherent physical uncertainty and macroscopic randomness. The current work illustrates that the above conclusion holds not only for Lorenz equation with three ODEs but also for Saltzman's equation with up-to nine ODEs, and thus has general meanings.

1 Introduction

It is well-known that, due to the sensitive dependence on initial condition (SDIC) of chaos [1], it is impossible to make accurate "long-term" prediction of chaotic dynamic systems. This is mainly because all

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numerical simulations always contain the so-called numerical noises, i.e. truncation and round-off errors. Thus, due to the SDIC of chaos, the local numerical noises propagate in a rather complicated way and are enlarged exponentially. It has been reported by a lots of researchers that numerical simulations of chaos based on the traditional Runge-Kutta method and double-precision data keep the same only in a small time interval but quickly become rather different for different numerical algorithms and thus are not reliable. Lorenz [2, 3] himself used low (i.e. 1st to 4th) order Taylor series method (TSM) and double-precision data to solve chaotic solutions of Lorenz equation and found that the chaotic numerical results are often sensitive not only to initial condition (i.e. the SDIC) but also to the numerical algorithms. Then, naturally, some negative comments about numerical simulations of chaos appeared, which suggested that "all chaotic responses are simply numerical noise and have nothing to do with the solutions of differential equations" [4]. Therefore, the reliability of chaotic numerical results becomes a serious problem.

The so-called Taylor series method (TSM) can be traced back up to the elegant works of Newton and Euler, and has been investigated by many researchers such as [5–9]. Currently, Barrio et al. [6] combined the TSM with the multiple precision (MP) library and parallel computation to develop an efficient software to solve coupled ODEs, which can automatically choose optimal time-step according to a given tolerance of equation.

Not knowing the works of Barrio et al. [5, 6] on the TSM, Liao [10–12] followed Lorenz's work to independently develope a numerical simulation method, namely the "clean numerical simulation" (CNS), so as to gain reliable, rather accurate results of chaotic dynamic systems in a finite but long enough interval of time. The CNS is based on *not only* the TSM *but also* a validation check of global reliability of results: it is superior to the TSM in reliablity, but more expensive in CPU time. Unlike Lorenz [3] who used the low-order TSM, Liao [10] employed rather high-order Taylor expansion to greatly decrease the truncation-error, and besides used the arbitrary-precision data of Mathematica to guarantee rather small round-off error. In this way, nunmereical simulations of chaos can be obtained with rather small tolerance of equation. However, a small tolerance of equation by oneself can *not* guarantee the reliability of chaotic results in a given interval: as reported by Liao [10], the chaotic numerical results of Lorenz equation given by the 60th-order TSM using the time-step $\Delta t = 10^{-2}$ and the 200-digit precision data have nearly the *same* maximum tolerance of equation at the level of 10^{-66} in the *whole* interval $[0, 1000]$, but is reliable only in the interval $[0, 180]$ that is much smaller than [0,1000]. This is mainly because the tolerance of equation is a local property, which, due to the SDIC of chaos, can not guarantee the global reliability of chaotic numerical simulations. To guarantee the global reliability of chaotic simulations in a given interval, Liao [10] suggested a validation check approach by proposing the concept "critical predictable time" T_c: a numerical simulation is reliable in a finite interval $[0, T_c]$. As suggested by Liao [10, 12], the critical prediction time T_c of a chaotic numerical simulation is determined by comparing it with those having smaller tolerances of equation. For example, for Lorenz equation solved by the CNS using $\Delta t = 10^{-2}$, the Mth-order TSM and the N-digit precision data, where $N = \max\{2M, 200\}$, Liao [10] gained the regression formula $T_c \sim 3M$ by means of the computer algebra software Mathematica. Then, according to this regression formula, Liao [10] obtained, for the first time, the reliable chaotic result in a long interval $[0, 1000]$, whose correction was validated by Wang et al. [13] using the CNS with the multiple precision and parallel computation, but much shorter CPU time. These reliable chaotic solutions of Lorenz equation in such a long interval have never been reported before. Note that Sarra and Meador [14] reported that there is a limit for the 16th-order TSM with the quad-double precision numbers. This indicates from the other side the importance of using arbitrary-precision data. In fact, using high enough order of TSM and accurate enough precision data, both Liao [10] and Wang et al [13] obtained convergent, reliable numerical solutions of chaos of Lorenz equation by means of the CNS. Thus, againest the negative comments [4] on numerical simulations of chaos, the works of Liao [10] and

Wang et al. [13] indicate the possibility of gaining reliable, convergent numerical simulations of chaos in a *finite* but long *enough* interval.

The CNS has been successfully applied to the famous Lorenz equation [10, 11] and the chaotic motion of stars orbiting in a plane about the galactic center [12]. In these works, the computer algebra system Mathematica was used to express data in arbitrary-precision. Currently, Wang et al [13] successfully gained reliable chaotic results of Lorenz equation in a rather long interval $0 \leq t \leq 2500$ by means of the CNS and parallel computation using multiple precision library (MP), which greatly improves the computational efficiency of the CNS.

Note that Lorenz equation is consist of three ordinary differential equations (ODEs), which is only a simplified model for the convection of fluid. In fact, based on Navier-Stokes equation and Boussinesq equation, Saltzman [15] studied a more complicated dynamic system with up-to nine ODEs for convection flow of fluid. In this article, we solve chaotic solutions of Saltzman's dynamic systems by means of the CNS with parallel computation and MP library. It is found that, for Saltzman's dynamic system with up to none ODEs, the micro-level inherent uncertainty also transfers into the observable, macroscopic randomness, due to the SDIC of chaos. Thus, this conclusion has general meanings.

2 Mathematical equations

Consider a two-dimensional convection flow between two free surfaces with a constant difference of temperature, governed by

$$\bigtriangledown^2 \frac{\partial \psi}{\partial t} + \frac{\partial(\psi, \bigtriangledown^2 \psi)}{\partial(x,z)} - \sigma \frac{\partial \theta}{\partial x} - \sigma \bigtriangledown^4 \psi = 0, \tag{1}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial(\psi, \theta)}{\partial(x,z)} - R\frac{\partial \psi}{\partial x} - \bigtriangledown^2 \theta = 0, \tag{2}$$

where ψ denotes the stream-function, θ the temperature, t the time, x, z the horizontal and vertical coordinate, σ the Prandtl Number, and R the Rayleigh Number, respectively. Here,

$$\frac{\partial(a,b)}{\partial(x,z)} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial z} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial z} \tag{3}$$

is the so-called Jacobian operator, and all variables are dimensionless. In 1962, using double-Fourier expansion, Saltzman [15] transformed the original Navier-Stokes equations (1) - (2) into an infinite number of nonlinear ordinary differential equations:

$$\begin{aligned} \dot{\Psi}(m,n) = & \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} C(m,n,p,q) \frac{\alpha^2(p,q)}{\alpha^2(m,n)} \Psi(p,q)\Psi(m-p,n-q) \\ & - \frac{(\sigma l m)i}{\alpha^2(m,n)} \Theta(m,n) - \sigma \alpha^2(m,n)\Psi(m,n), \end{aligned} \tag{4}$$

$$\begin{aligned} \dot{\Theta}(m,n) = & \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} C(m,n,p,q) \Psi(p,q)\Theta(m-p,n-q) \\ & + (R\, l\, m)i\Psi(m,n) - \alpha^2(m,n)\Theta(m,n) \end{aligned} \tag{5}$$

where $\Psi(m,n), \Theta(m,n)$ are coefficients of Fourier series of ψ and θ, the dot denotes the differentiation with respect to the time, $i = \sqrt{-1}$, $l = 2\pi H/L$, $h = \pi$,

$$C(m,n,p,q) = lh(mq - np), \quad \alpha^2(a,b) = (2\pi H/L)^2 a^2 + \pi^2 b^2,$$

H the height and L the wave-length, m and n the wave number in the x and y-direction, respectively.

For the sake of simplicity, Saltzman [15] chose the seven components

$$\Psi_1(3,1),\ \Psi_1(4,1),\ \Psi_1(1,2),\ \Theta_2(3,1),\ \Theta_2(4,1),\ \Theta_2(1,2),\ \Theta_2(0,2)$$

to gain a highly truncated dynamic system consist of the seven ODEs

$$\begin{cases} \dot{A} &= 23.521BC - 1.500D - 148.046A, \\ \dot{B} &= -22.030AC - 1.589E - 186.429B, \\ \dot{C} &= 1.561AB - 0.185F - 400.276C, \\ \dot{D} &= -16.284CE - 16.284BF - 13.958AG - 1460.631\lambda A \\ &\quad -14.805D, \\ \dot{E} &= 16.284CD - 16.284AF - 18.610BG - 1947.508\lambda B \\ &\quad -18.643E, \\ \dot{F} &= 16.284AE + 16.284BD - 486.877\lambda C - 40.028F, \\ \dot{G} &= 27.916AD + 37.220BE - 39.479G, \end{cases} \tag{6}$$

where $\lambda = R/R_c$ is a physical parameter, $A = \Psi_1(3,1)$ and $D = \Theta_2(3,1)$ represent the cellular streamline and thermal fields for the Rayleigh critical mode, $G = \Theta_2(0,2)$ the departure of the vertical temperature, with the definitions

$$B = \Psi_1(4,1),\ C = \Psi_1(1,2),\ E = \Theta_2(4,1),\ F = \Theta_2(1,2),$$

respectively.

As pointed out by Saltzman [15], in certain cases, A, D, G of (6) fluctuate but others eventually tend to zero. Following Saltzman's work [15], Lorenz [1] put forward the famous Lorenz equations, which can describe the so-called chaotic convection with only three very concise equations, although it is a rough approximation of the N-S equation.

Neglecting B, C, E, F in Saltzman's dynamic system (6), one has a dynamic system with three degree of freedom

$$\begin{cases} \dot{A} &= -148.046A - 1.500D, \\ \dot{D} &= -13.958AG - 1460.631\lambda A - 14.805D, \\ \dot{G} &= 27.916AD - 39.479G. \end{cases} \tag{7}$$

It is rather similar to the famous Lorenz equation. Similarly, one has a dynamic system with four degree of freedom:

$$\begin{cases} \dot{A} &= -1.500D - 148.046A, \\ \dot{B} &= -186.429B, \\ \dot{D} &= -13.958AG - 1460.631\lambda A - 14.805D, \\ \dot{G} &= 27.916AD - 39.479G, \end{cases} \tag{8}$$

a dynamic system with five degree of freedom

$$\begin{cases} \dot{A} &= 23.521BC - 1.500D - 148.046A, \\ \dot{B} &= -22.030AC - 186.429B, \\ \dot{C} &= 1.561AB - 400.276C, \\ \dot{D} &= -13.958AG - 1460.631\lambda A - 14.805D, \\ \dot{G} &= 27.916AD - 39.479G, \end{cases} \tag{9}$$

and a dynamic system with six degree of freedom

$$\begin{cases} \dot{A} &= 23.521BC - 1.500D - 148.046A, \\ \dot{B} &= -22.030AC - 1.589E - 186.429B, \\ \dot{C} &= 1.561AB - 400.276C, \\ \dot{D} &= -16.284CE - 13.958AG - 1460.631\lambda A - 14.805D, \\ \dot{E} &= 16.284CD - 18.610BG - 1947.508\lambda B - 18.643E, \\ \dot{G} &= 27.916AD + 37.220BE - 39.479G, \end{cases} \tag{10}$$

respectively. All of them are chaotic in some cases. In this article, we investigate these chaotic dynamic systems from a physical and statistical viewpoint by means of the Clean Numerical Simulation (CNS) in a similar way suggested by Liao [11, 12]. However, different from Liao [11, 12], we employ the multiple precision (MP) library and parallel computation in stead of Mathematica so as to greatly increase the computation efficiency. Our computations indicate that Liao's conclusion reported in [11, 12] has general meaning: due to the SDIC of chaos, the micro-level inherent physical uncertainty of a dynamic system might indeed transfer into the macroscopic randomness so that chaos is a bridge between the micro-level uncertainty and macroscopic randomness.

3 The CNS approach with multiple precision and parallel computation

For simplicity, let us use the dynamic system (7) with the three degree of freedom to briefly describe the basic ideas of the CNS and to show its validity. Assume that its solution is infinitely differentiable so that its Taylor series exists in the whole interval.

As mentioend before, the CNS is based on the TSM and the validation chech of reliability. Firs of all, the Mth-order truncated Taylor expansion in the interval $t \in [t_n, t_n + \tau]$ read

$$A(t) = A(t_n) + \sum_{k=1}^{M} \alpha_k (t - t_n)^k, \tag{11}$$

$$D(t) = D(t_n) + \sum_{k=1}^{M} \beta_k (t - t_n)^k, \tag{12}$$

$$G(t) = G(t_n) + \sum_{k=1}^{M} \gamma_k (t - t_n)^k, \tag{13}$$

where M denotes the order of Taylor expansion, and

$$\alpha_k = \frac{1}{k!}\frac{d^k A}{dt^k},\ \beta_k = \frac{1}{k!}\frac{d^k D}{dt^k},\ \gamma_k = \frac{1}{k!}\frac{d^k G}{dt^k},\ \text{at } t = t_n \tag{14}$$

are the coefficients. Substituting the Taylor expansions (11)–(13) into (7) gives us the recursion formula

$$\alpha_{k+1} = -\frac{1}{k+1}\left(148.046\alpha_k + 1.500\beta_k\right), \tag{15}$$

$$\beta_{k+1} = -\frac{1}{k+1}[13.958\sum_{i=0}^{k} \alpha_{k-i}\gamma_i + 1460.631\lambda\,\alpha_k + 14.805\beta_k], \tag{16}$$

$$\gamma_{k+1} = \frac{1}{k+1}[27.916\sum_{i=0}^{k} \alpha_{k-i}\beta_i - 39.479\gamma_k]. \tag{17}$$

Obviously, the larger the order M of Taylor expansions (11)–(13), the smaller the truncation error. Besides, in order to greatly decrease the round-off error, we should use high precision data. For this purpose, we use the GMP (GNU Multiple Precision Arithmetic Library)[a], which is a free library for arbitrary-precision arithmetic, operating on signed integers, rational numbers, and floating point numbers. Note that there are no practical limits to the precision except the ones implied by the available memory in the machine GMP runs on. Therefore, using accurate enough data in GMP, we can control the round-off error to the required level. Thus, using the high enough order of Taylor expansion and accurate enough GMP data, we can control the numerical noises (i.e. the truncation and round-off error) within a required level, as shown below.

Like Wang et al. [13], the parallel computation is used to increase the computation efficiency of the CNS approach. The Message Passing Interface (MPI) is widely used for parallel computation in double precision. In 2008, Tomonori Kouya developed a package MPIGMP, based on which one can use MPI freely and pass multiple precision data directly. For more details about parallel computation and multiple precision, please refer to Nikolaevskaya, Khimich, and Chistyakova [16]. Unlike Liao [10–12] who used Mathematica to guarantee high precision of data, we use here the MPIGMP with parallel computation. For simplicity, all data are expressed in the K-digit precision by means of GMP, where M denotes the order of Taylor expansion and $K = \max\{2M, 200\}$.

Obviously, the terms $\sum_{i=0}^{k} \alpha_{k-i}\gamma_i$ in Eq. (16) and $\sum_{i=0}^{k} \alpha_{k-i}\beta_i$ in Eq. (17) spend most of the CPU times. For parallel computation of these two terms, we allocate the task to N processes and then add the results in each process to the main process. Then, the result of the summation is broadcasted to all of the N processes from the main process. By means of parallel computation in this way, the CPU time can be decreased greatly, as shown in Fig. 1.

Unlike the TSM, the CNS is additionally based on a kind of validation check of reliability of results. Generally speaking, the CNS provides us a *practical* way to gain reliable, *accurate enough* solutions of chaotic dynamic systems with *high enough* precision in a *finite* but *long enough* time interval. The key point of the CNS is to provide chaotic solution in the so-called critical predictable time T_c: numerical simulations are reliable in the interval $0 \le t \le T_c$ but might decouple when $t > T_c$. As suggested by Liao [10–12], the critical predictable time T_c is determined by the decoupling time of two numerical simulations given by two different algorithms. For details, please refer to Liao [10–12]. It should be emphasized that, due to the SDIC of chaos, the numerical noises enlarge exponentially. Since the two chaotic results agree well in $0 \le t \le T_c$, the corresponding numerical noises must be negligible.

As mentioned above, the validation check of global reliablity is a key point of the CNS, which often provides a regression relationship between T_c and M for a given time-step Δt, where M is the order of TSM. For example, we use here the MPIGMP data in the K-digit precision, where $K = \max\{2M, 200\}$ and M is the order of Taylor expansion, respectively. In the case of $\Delta t = 1/200$, it is found that

$$T_c \approx 0.0423M + 0.3676, \tag{18}$$

as shown in Fig. 2. Thus, in order to gain reliable chaotic results in the intervals $0 \le t \le 4$ and $0 \le t \le 10$ by means of $\Delta t = 1/200$, we should use the 100th and 240th-order Taylor expansion with the corresponding GMP data in the 200 and 480-digit precision, respectively. In this way, the reliability of the CNS approach is under control. It should be emphasized that regression formulas like (18) plays a very important role in the CNS: it guarantees the global reliablity of chaotic numerical results.

[a]For details about GMP, please visit the website http://gmplib.org/manual/

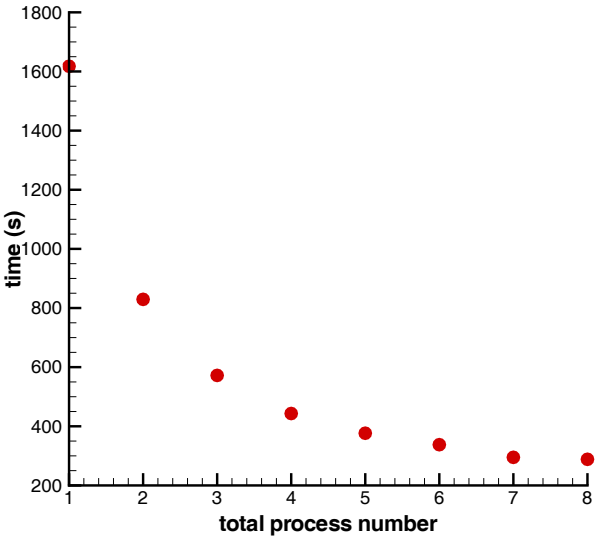

Fig. 1 The CPU time versus the number N of used processes for the system (7) of three freedom-degree in the interval $0 \leq t \leq 10$ by means of $M = 240, K = 480$ and $\Delta t = 1/200$.

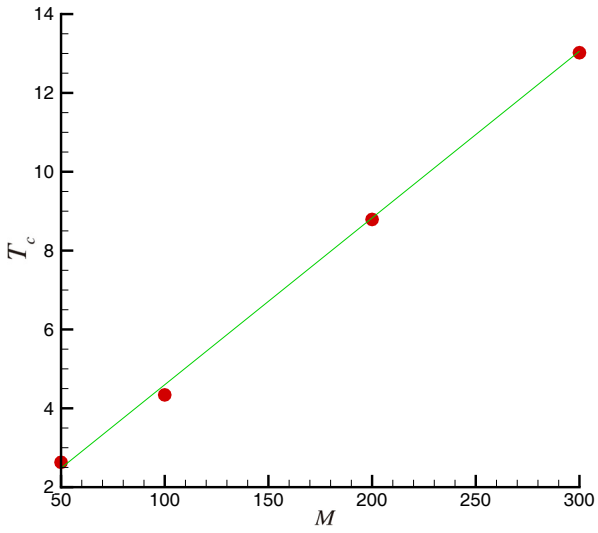

Fig. 2 The critical predictable time T_c versus the order M of Taylor expansion for the system (7) in case of $A(0) = 1, D(0) = 0.001, G(0) = 0.001$ with $\Delta t = 1/200$. Symbols: computed results; Solid line: $T_c = 0.0423M + 0.3676$.

4 Propagation of inherent physical micro-level uncertainty

As pointed out by Liao [11], Lorenz equation is a simplified model based on Navier-Stokes equations and Boussinesq approximation. Its basic dependent variables are temperature and velocity of fluid. Note that

temperature is a concept of statistics. Besides, due to the continuum-assumption of fluid, the velocity of fluid is a concept of statistics as well. However, it is well-known that all statistic quantities contain the so-called statistic fluctuation. Thus, the statistic fluctuations of velocity and temperature of fluid are physically *inherent* and thus *unavoidable* in essence. So, strictly speaking, the initial condition of Lorenz equation involves the physical uncertainty. It should be emphasized that such kind of inherent physical uncertainty has *nothing* to do with the measurement ability of human being! As pointed out by Liao [11], such kind of inherent physical uncertainty of velocity of fluid might be in the level of 10^{-30}, which is much smaller than measured velocity and measurement error. Fortunately, by means of the CNS, the numerical noises can be decreased to a level much smaller than 10^{-30}. For example, Liao [11] investigated the propagation of the micro-level physical inherent uncertainty of initial conditions of Lorenz equation and found that, due to SDIC of chaos, such kind of micro-level physical uncertainty transfers into the macroscopic randomness in a long enough time interval.

Note that, like Lorenz equation, Saltzman's equations are also based on Navier-Stokes equations and Boussinesq approximation. Thus, like Lorenz equation, the initial conditions of Saltzman's equations also contain inherent physical micro-level uncertainty as well. As pointed out by Liao [11], such kind of uncertainty could be in the level of 10^{-30}, but is enlarged exponentially due to the SDIC of chaos. This is indeed true, as shown below.

For example, let us first consider Saltzman's equation with three variables $A(t), D(t)$ and $G(t)$. Note that $A(t)$ is consist of two parts: the mean value $A^*(t)$, which is observable or measured, and the inherent physical micro-level uncertainty $\tilde{A}(t)$ of statistical fluctuation, say,

$$A(t) = A^*(t) + \tilde{A}(t), \tag{19}$$

where the physical micro-level uncertainty $\tilde{A}(t)$ of statistical fluctuation is much smaller than the observable term $A^*(t)$. So do

$$D(t) = D^*(t) + \tilde{D}(t), \quad G(t) = G^*(t) + \tilde{G}(t).$$

Without loss of generality, consider the case of the observable initial conditions

$$A^*(0) = 1, \ D^*(0) = G^*(0) = 0.001, \tag{20}$$

which are assumed to be exact. According to the central limit theorem in probability theory, we assume that, at $t = 0$, the statistical fluctuations of velocity and temperature of fluid are in the normal distribution with zero mean and a micro-level deviation σ_0, i.e.

$$\left\langle \tilde{A} \right\rangle = \left\langle \tilde{D} \right\rangle = \left\langle \tilde{G} \right\rangle = 0, \ \left\langle \tilde{A}^2 \right\rangle = \left\langle \tilde{D}^2 \right\rangle = \left\langle \tilde{G}^2 \right\rangle = \sigma_0, \ \text{at } t = 0, \tag{21}$$

where $\sigma_0 = 10^{-30}$ is chosen in this article.

According to the regression formula (18), in case of $\Delta t = 1/200$, we had to use the 240th-order Taylor expansion and GMP data in accuracy of 480-digit precision so as to gain reliable chaotic simulations of the system (7) in the interval $0 \le t \le 10$. We use 2000 different initial conditions: each initial condition with the exact observable part (20) and the micro-level uncertainty (21) of statistical fluctuation gives us a reliable but different chaotic result in the interval $0 \le t \le 10$. Let $\langle A(t) \rangle, \langle D(t) \rangle, \langle G(t) \rangle$ and $\sigma_A(t), \sigma_D(t), \sigma_G(t)$ denote the sample mean and unbiased estimate of standard deviation of the 2000 reliable chaotic results of $A(t)$, $D(t)$ and $G(t)$, respectively. Define the so-called uncertainty intensity

$$\varepsilon(t) = \sqrt{\frac{\sigma_A^2(t) + \sigma_D^2(t) + \sigma_G^2(t)}{\langle A(t) \rangle^2 + \langle D(t) \rangle^2 + \langle G(t) \rangle^2}}. \tag{22}$$

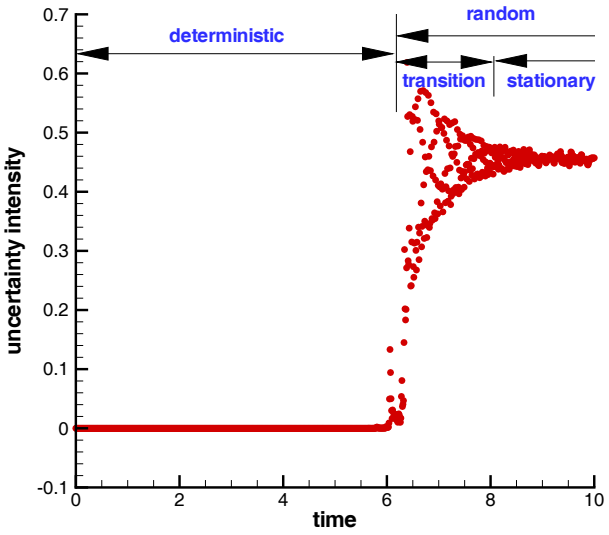

Fig. 3 The uncertainty intensity $\varepsilon(t)$ of (7) in $0 \leq t \leq 10$ given by the 2000 different initial conditions with the mean value (20) and the micro-level uncertainty (21) of statistical fluctuation in the case of $M = 240$, $K = 480$ and $\Delta t = 1/200$.

As shown in Fig. 3, there exists a time interval $t \in (0, T_d)$ in which the uncertainty intensity $\varepsilon(t)$ is so tiny that the dynamic system looks like deterministic and the influence of the micro-level inherent physical uncertainty of initial condition is negligible. However, when $t > T_d$, the uncertainty intensity becomes sensitive to the statistical fluctuation of initial conditions so that the dynamic system looks like random. In other words, due to the SDIC of chaos, the micro-level inherent physical uncertainty of the temperature and velocity of initial condition transfers into the observable, macroscopic randomness in a long enough time interval. Note that when $t > 8$, the uncertainty intensity $\varepsilon(t)$ tends to be independent of time, corresponding to the steady state of system (7) in the meaning of statistics.

For systems with 4 to 7 degrees of freedom, we can investigate the influence of the micro-level inherent physical uncertainty of initial conditions on the chaotic results in a similar way by means of the parallel CNS. Note that, when $t > 8$, the uncertainty intensity $\varepsilon(t)$ tends to be independent of time in all cases, corresponding to a steady state in the meaning of statistics. Note also that, as the degree of freedom of the system increases, the steady-state uncertainty intensity (when $t > 8$) decreases, as shown in Fig. 4. It is found that, in all cases, the micro-level inherent physical uncertainty of initial conditions transfers into macroscopic randomness in a long enough time interval, as shown in Fig. 4. This suggests once again that chaos might be an origin of some macroscopic randomness. This also indicates that this conclusion has general meanings as well. Note that Saltzman's model is closely related to convection flows governed by the N-S equations.

5 Conclusions and discussions

In this paper, the so-called "Clean Numerical Simulation" (CNS) is applied to gain reliable chaotic solutions of Saltzman's dynamic system, which is a simplified model for convection flow of fluid. Based on the high-

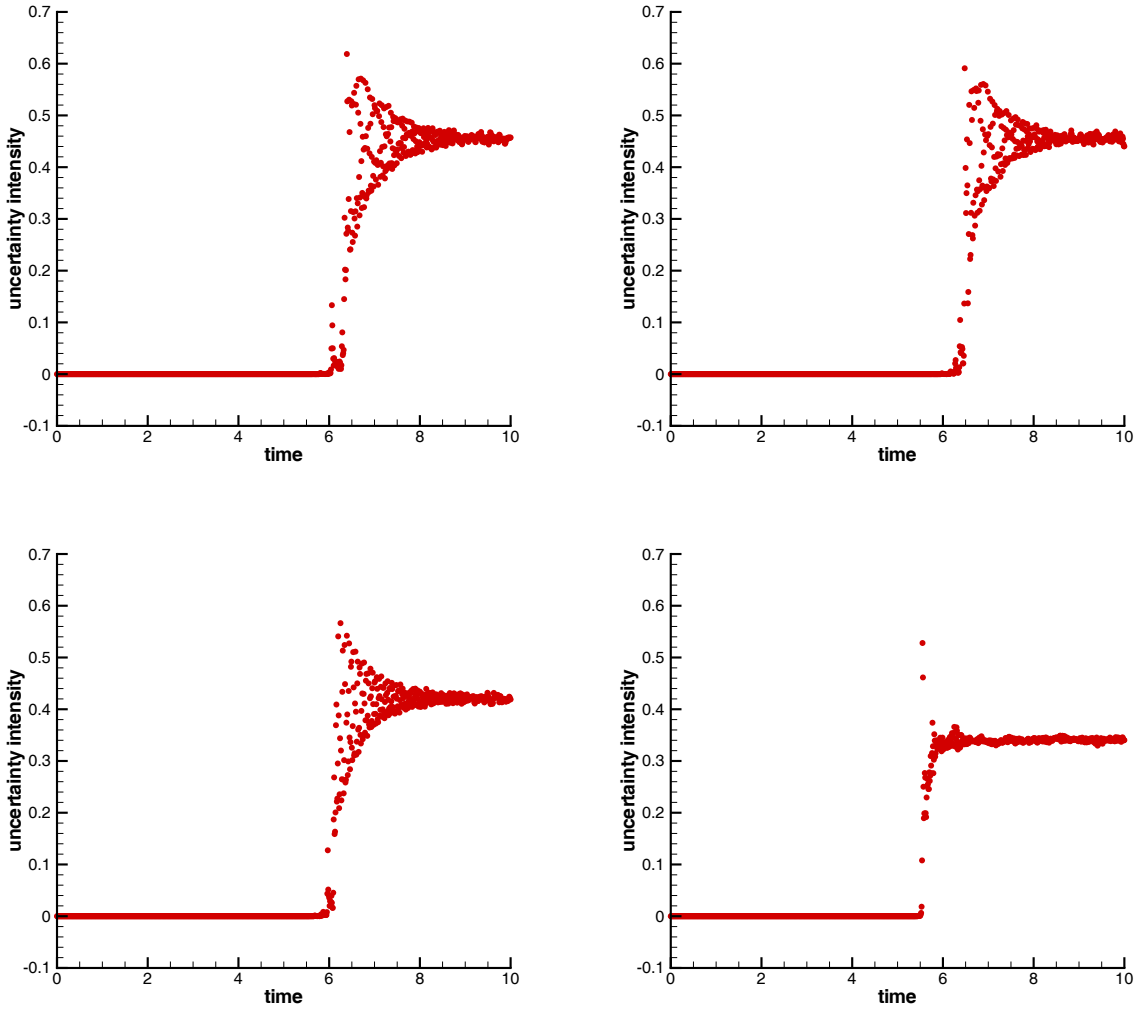

Fig. 4 The uncertainty intensity $\varepsilon(t)$ of chaotic dynamic systems (8) to (10) and (6) in the interval $0 \leq t \leq 10$ given by the CNS with the 2000 different initial conditions in the normal distribution with zero mean and micro-level deviation $\sigma = 10^{-30}$. Top-left: system (8); Top-right: system (9); Bottom-left: system (10); Bottom-right: system (6).

order TSM (with multiple precision and parallel computation) and a validation check of global reliability of results, the CNS provides us a practical way to gain reliable, accurate enough solutions of chaotic dynamic systems in a finite but long enough time interval. The numerical noises of the CNS can be controlled to be so small that even the influence of the micro-level inherent physical uncertainty of initial conditions can be investigated accurately. The use of the parallel computation can greatly increase the computational efficiency. It is found that the micro-level inherent physical uncertainty of initial conditions of chaotic Saltzman's system transfers into macroscopic randomness. This suggests that chaos might be a bridge between micro-level physical uncertainty and macroscopic randomness. This article indicates that the above conclusion holds not only for Lorenz equation with three ODEs [10, 11] but also for Saltzman's equation with up-to nine ODEs, and thus has general meanings.

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