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SHORT COMMUNICATION

A new branch of the temperature distribution of boundary-layer flows over an impermeable stretching plate

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Abstract In 2005 it was reported that the boundary-layer flows over an impermeable stretching plate have a new branch of solutions. In this short communication, the corresponding heat transfer problem is considered, and a new branch of temperature distribution is obtained. It is found that the new branch of temperature distributions is mostly rather close to the known branch of solutions, except in case of small Prandtl number. Thus, it is practically rather hard to distinguish the two branches of temperature distributions.

The boundary-layer flow of the incompressible Newtonian viscous fluid over a stretching sheet is an important type of flows occurring in a number of engineering processes, such as the aerodynamic extrusion of plastic sheets, the boundary layer along liquid film condensation process, the cooling process of metallic plate in a cooling bath, and glass and polymer industries. Since the pioneering work of Sakiadis [1, 2], various aspects of the problem were investigated by many authors. Tsou [3] considered the problem with constant surface velocity and temperature. Crane [4], Vleggaar [5], and Gupta and Gupta [6] analyzed the stretching problem with constant surface temperature, while Soundalgekar and Ramana Murty [7] investigated the constant surface velocity case with power law temperature variation. Grubka and Bobba [8], and Chen and Char [9] extend this problem to the case the heat transfer occurring on a linear impermeable stretched surface with a power law temperature.

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The laminar thermal boundary-layer viscous flow over a stretching impermeable plate can be described by the continuity, momentum and energy equations [8, 9]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{1b}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{1c}$$

subject to the boundary conditions:

$$u=a(x+b)^{\lambda} \quad v=0 \quad T=T_\infty+C(x+b)^k \quad \text{at} \quad y=0 \tag{1d}$$

$$u\to 0, \quad T\to T_\infty, \quad \text{as} \quad y\to+\infty, \tag{1e}$$

where the x-axis runs along the surface in the direction of motion, the y-axis is perpendicular to it, u and v are the velocity components of the fluid in the x and y directions, T denotes the temperature distribution, T_∞ the temperature at infinity, α the thermal diffusivity, ν the kinematic viscosity coefficient, a, b, λ, C and k are constant coefficients, respectively. Let ψ denote the stream function. Using the similarity transformations

$$\psi=a\sqrt{\frac{\nu}{a(1+\lambda)}}(x+b)^{\frac{\lambda+1}{2}}F(\xi),$$

$$\xi=\sqrt{\frac{a(1+\lambda)}{\nu}}(x+b)^{\frac{\lambda-1}{2}}y,$$

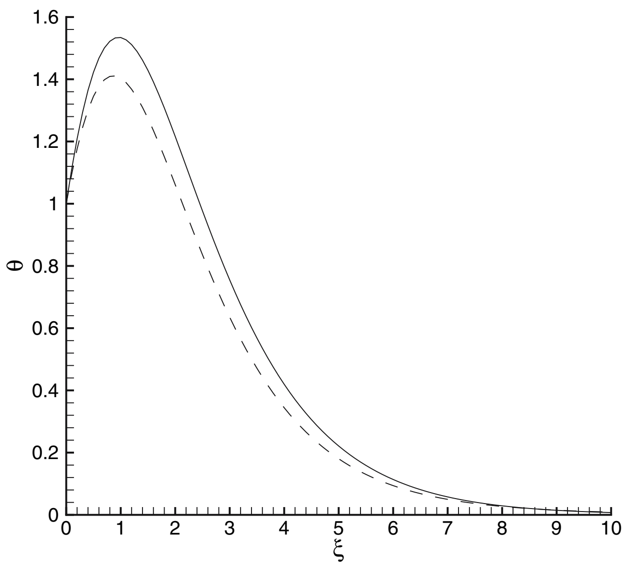

Fig. 1 Dual solutions when $Pr = 1$, $\beta = 0.6$ and $\kappa = -3$. *Solid line* first branch of solutions, *dashed line* second branch of solutions

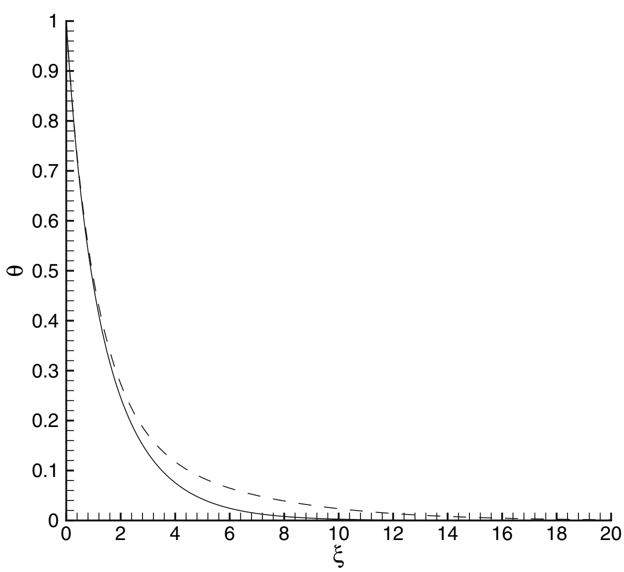

Fig. 2 Dual solutions when $Pr = 1$, $\beta = 2$ and $\kappa = -1$. *Solid line* first branch of solutions, *dashed line* second branch of solutions

$$\theta = \frac{T - T_\infty}{T_w - T_\infty},$$

where $a \neq 0$ and $a(1 + \lambda) > 0$, Eqs. (1a)–(1c) become

$$F'''(\xi) + \frac{1}{2}F(\xi)F''(\xi) - \beta F'^2(\xi) = 0, \quad (2a)$$

$$\theta''(\xi) + \frac{1}{2}Pr F(\xi)\theta'(\xi) - \kappa Pr(1-\beta)F'(\xi)\theta(\xi) = 0, \quad (2b)$$

subject to the boundary conditions:

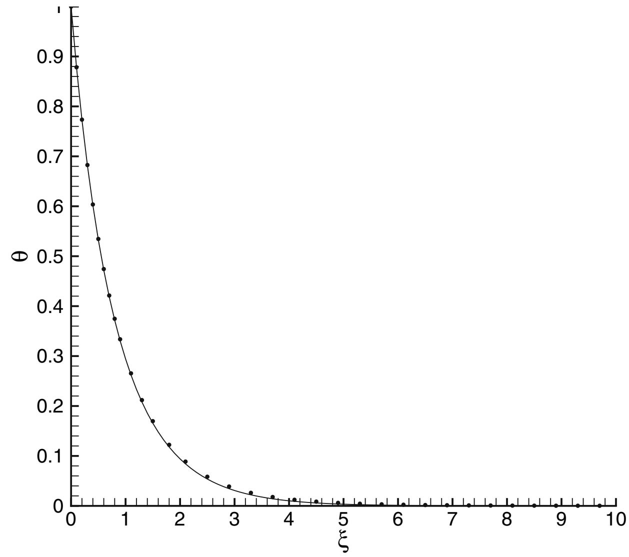

Fig. 3 Dual solutions when $Pr = 2$, $\beta = 2$ and $\kappa = -1$. *Solid line* first branch of solutions, *symbols* second branch of solutions

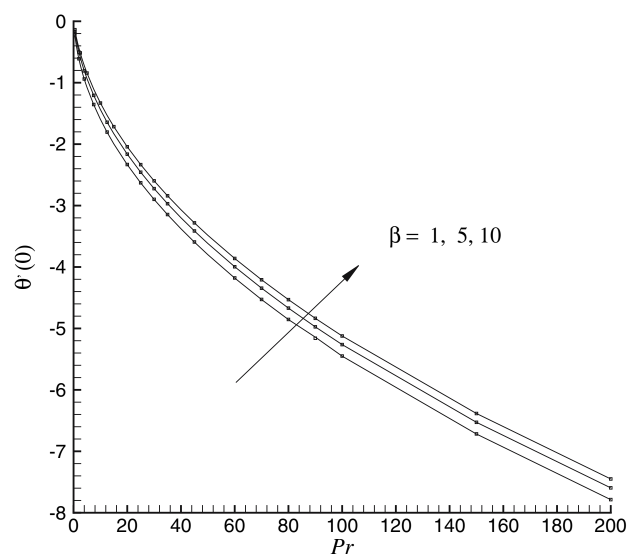

Fig. 4 The comparison of $\theta'(0)$ of the two branches of solutions for various Pr when $\kappa = 0$. *Line* first branch of solutions, *symbol* second branch of solutions

$$F(0) = 0 \quad F'(0) = 1 \quad F'(+\infty) = 0, \quad (2c)$$

$$\theta(0) = 1 \quad \theta(+\infty) = 0, \quad (2d)$$

where $Pr = \nu/\alpha$ is the Prandtl number, and

$$\beta = \frac{\lambda}{1+\lambda}$$

Recently, by means of a new analytic method, namely the homotopy analysis method (HAM) [10–18], Liao [19]

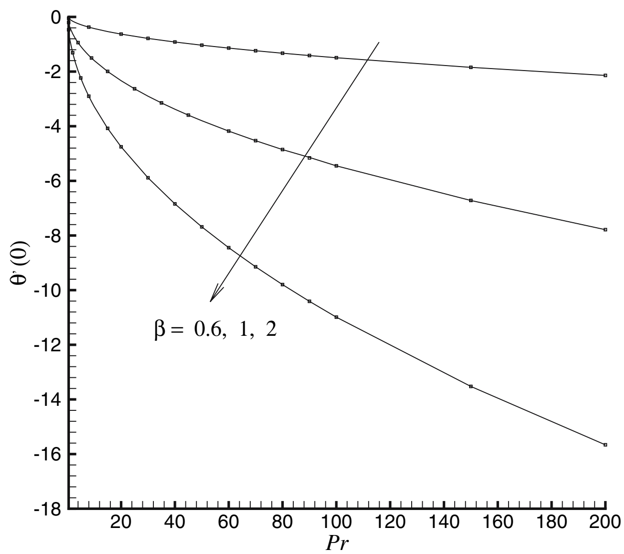

Fig. 5 The comparison of $\theta'(0)$ of the two branches of solutions for various Pr when $\kappa = -1$. *Line* first branch of solutions, *symbol* second branch of solutions

found a new branch of solutions of the boundary-layer flows over a stretching impermeable plate governed by Eqs. (2a) and (2c). The two branches of solutions are so close that the new one has never been reported even by means of numerical techniques. Obviously, for each solution $F(\xi)$ of Eqs. (2a) and (2c), there exists a corresponding temperature distribution $\theta(\xi)$ of the linear equation (2b) with the linear boundary conditions (2d).

It is very easy to solve Eqs. (2b) and (2d). Here, we use two different methods. First, by means of the HAM [10–15], we obtain the series solution of $\theta(\xi)$ for each branch of solutions of $F(\xi)$. To shorten the length of this communication, all mathematical expressions are neglected here. Second, using the accurate HAM results as the initial guess, we apply the Keller-box numerical method to get convergent numerical solutions. All of our series solutions and numerical results agree very well.

It is found that the temperature distributions related to the new branch of fluid flows found by Liao [19] have been never reported. Besides, in many cases, it is rather hard to distinguish the two branches of temperature distributions. In summary, our main conclusions are:

1. Generally speaking, for small values of Pr, the two branches of solutions have obvious differences, as shown in Figs. 1 and 2. However, for $Pr \geq 2$, the difference becomes hard to distinguish, as shown in Fig. 3. This may be the reason why the second branch of temperature distribution has never been reported even by numerical methods.

Table 1 $\theta'(0)$ in case of $\kappa = 0$ and $\beta = 1$

Pr	First branch	Second branch	Relative difference (%)
0.5	–0.233694	–0.203089	13.0962
1.0	–0.388657	–0.374936	3.5304
1.5	–0.511590	–0.503155	1.6488
2.0	–0.616400	–0.610110	1.0204
2.5	–0.709239	–0.704066	0.7294
3.0	–0.793435	–0.788934	0.5673
3.5	–0.871018	–0.866970	0.4647
4.0	–0.943332	–0.939609	0.3947
4.5	–1.011321	–1.007837	0.3445
5.0	–1.075677	–1.072379	0.3066
7.5	–1.357945	–1.355192	0.2027
10.0	–1.596237	–1.593749	0.1559
12.5	–1.806336	–1.804000	0.1293
15.0	–1.996362	–1.994133	0.1117
20.0	–2.333884	–2.331807	0.0890
25.0	–2.631374	–2.629371	0.0761
30.0	–2.900353	–2.898429	0.0663
35.0	–3.147778	–3.145879	0.0603
40.0	–3.378064	–3.376216	0.0547
45.0	–3.594367	–3.592567	0.0501
50.0	–3.799030	–3.792567	0.0480
60.0	–4.179623	–4.177841	0.0426
70.0	–4.529652	–4.527888	0.0389
80.0	–4.855463	–4.853720	0.0359
90.0	–5.161480	–5.159755	0.0334
100.0	–5.450932	–5.449216	0.0315
150.0	–6.718631	–6.717000	0.0243
200.0	–7.787474	–7.785838	0.0210
500.0	–12.423930	–12.422359	0.0126
1,000.0	–17.649331	–17.647788	0.0087
2,000.0	–25.039290	–25.037766	0.0061

2. The wall temperature gradient $\theta'(0)$ has important physical meaning. The curves of wall temperature gradients $\theta'(0)$ versus Pr for some values of β at $\kappa = 0$ and $\kappa = -1$ are as shown in Figs. 4 and 5. It should be emphasized that, for most values of Pr, the differences of $\theta'(0)$ of the two branches of solutions are so small that it is even hard to distinguish them, as clearly shown in Figs. 4 and 5. For example, the relative differences of $\theta'(0)$ when $\beta = 1$ and $\kappa = 0$ are 3.53, 0.307, 0.156, 0.0315 and 0.0087% for $Pr = 1, 5, 10,$ 100 and 1,000, respectively, as shown in Table 1. When $\kappa = -1$ and $\beta = 2$, the relative differences of $\theta'(0)$ for $Pr = 1, 5, 10, 100$ and 1,000 are 1.70, 0.064, 0.032, 0.0064 and 0.0018% respectively, which are correspondingly even smaller than those in case of

Table 2 $\theta'(0)$ in case of $\kappa = -1$ and $\beta = 2$

Pr	First branch	Second branch	Relative difference (%)
0.5	−0.532231	−0.468889	13.5090
1.0	−0.852605	−0.838356	1.6996
2.0	−1.313661	−1.310231	0.2618
3.0	−1.669588	−1.667473	0.1268
4.0	−1.970326	−1.968669	0.0842
5.0	−2.235603	−2.234175	0.0639
6.0	−2.475613	−2.474323	0.0521
7.0	−2.696440	−2.695243	0.0444
8.0	−2.902059	−2.900928	0.0390
9.0	−3.095237	−3.094157	0.0349
10.0	−3.277991	−3.276951	0.0317
15.0	−4.078727	−4.077802	0.0227
20.0	−4.754076	−4.753209	0.0182
30.0	−5.887341	−5.886534	0.0137
40.0	−6.842938	−6.842164	0.0113
50.0	−7.684938	−7.684184	0.0098
60.0	−8.446222	−8.445482	0.0088
70.0	−9.146332	−9.145603	0.0080
80.0	−9.798003	−9.797282	0.0074
90.0	−10.410084	−10.409370	0.0069
100.0	−10.989018	−10.988309	0.0064
150.0	−13.524608	−13.523918	0.0051
200.0	−15.662304	−15.661625	0.0043
500.0	−24.935394	−24.934738	0.0026
1,000.0	−35.386271	−35.385631	0.0018

$\kappa = 0$ and $\beta = 1$. In some cases, the difference is so small that it is hard to distinguish the two branches of temperature distributions even by means of numerical methods (Table 2).

Therefore, although there exists two branches of different heat transfer profiles of boundary-layer flows over an impermeable stretching sheet, the two branches of temperature distributions are close to each other, except for some small values of *Pr* number. The wall-gradients of the new branch of temperature distributions are in most cases rather close to those of the known ones, and thus hard to be distinguished each other. Physically, such small differences of two branches of temperature distributions may not bring any observed difference for us in practice. However, it is not clear if such situation implies the instability of the related boundary layer flows.

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