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International Journal of Non-Linear Mechanics 38 (2003) 1173–1183
INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS
www.elsevier.com/locate/ijnonlinmec

An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude

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Abstract

The currently developed analytic technique known as the homotopy analysis method is employed to propose a new approach for free oscillations of positively damped systems with algebraically decaying amplitude. In contrast to perturbation techniques, this approach is valid even for damped systems without any small/large parameters. Besides, unlike other analytic techniques, this approach itself provides us with a convenient way to adjust and control convergence of approximation series. Some typical examples are employed to illustrate its validity, effectiveness and flexibility. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Non-linear oscillation; Positively damped systems

1. Introduction

Perturbation method [1–11] is one of the most widely applied analytic tools for non-linear problems. By means of perturbation techniques, a lot of important properties and interesting phenomena of non-linear problems have been revealed. Recently, the singular perturbation techniques are considered among the top 10 progress of the theoretical and applied mechanics in the 20th century [12]. So, it is out of question that perturbation techniques play important roles in the development of science and engineering.

Essentially, perturbation techniques are based on the existence of a small/large parameter or variable, which is often called *perturbation quantity*. Obviously, the existence of perturbation quantities is a cornerstone of perturbation techniques. However, it is perturbation quantity which brings perturbation techniques some restrictions. First, it is impossible that *each* non-linear problem contains such a perturbation quantity. Secondly, even if perturbation quantities exist, perturbation techniques might fail to give a satisfactory result. For example, both straightforward perturbation method and singular perturbation technique failed to give a satisfactory theoretical drag formula of a sphere in a uniform stream [13]. This is mainly because, like other analytic methods, perturbation techniques *themselves* cannot provide us with a convenient way to *adjust* and *control* convergence regions of series of approximations.

The dependence of perturbation techniques on small/large parameters might be avoided by introducing a so-called artificial small parameter. In 1892 Lyapunov [14] considered the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x,$$

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PII: S0020-7462(02)00062-8

where $A(t)$ is a time periodic matrix. Lyapunov [14] introduced an artificial parameter m to replace this equation with the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = mA(t)x$$

and then calculated power series expansions over m for the solutions. In many cases Lyapunov proved that series converge for $m = 1$, and therefore one can put in the final expression by setting $m = 1$. The above approach is called Lyapunov's artificial small parameter method [14]. This idea was further employed by some scientists such as Karmishin et al. [15], who developed the so-called δ-expansion method. However, both the artificial parameter method and δ-expansion method need a principal *rule* to determine the place where the artificial parameter or δ should appear. However, to the best of our knowledge, such a kind of rule has not been reported. Besides, like perturbation techniques, both the artificial small parameter method and the δ-expansion method *themselves* do not provide us with a convenient way to *adjust* and *control* convergence regions of series of analytic approximations.

The idea of artificial parameter can be generally described by the homotopy [16], a concept of topology [17]. Based on the homotopy, many numerical techniques, such as the continuation method [18] and the homotopy continuation method [19], were developed.

In summary, to the best of the author's knowledge, neither perturbation techniques nor artificial small parameter methods nor δ-expansion method provide us with a convenient way to *adjust* and *control* convergence regions of series of analytic approximations. Besides, the efficiency of approximating a non-linear problem has not been taken into enough account. So, it is necessary to develop some new, more efficient analytic methods.

A kind of analytic method, namely the homotopy analysis method [20–27,13], was proposed by introducing an auxiliary parameter $\hbar$ to construct a new kind of homotopy in a more general form. The homotopy analysis method has the following advantages:

1. It is valid even if a given non-linear problem does *not* contain any small/large parameters *at all*;
2. It *itself* can provide us with a convenient way to *adjust* and *control* convergence regions of series of analytic approximations.
3. It can be employed to *efficiently* approximate a non-linear problem by *choosing* different sets of base functions.

In this paper the homotopy analysis method is further employed to free oscillations of positively damped systems with algebraically decaying amplitude, governed by

$$\ddot{u}(t) = f[\dot{u}(t), u(t)], \quad t \geqslant 0 \tag{1}$$

and subject to the initial conditions

$$u(0) = \alpha, \quad \dot{u}(0) = \beta. \tag{2}$$

Physically, a free oscillation of a positively damped system has two different time scales. One is related to the frequency, the other to the decaying of the amplitude of the oscillation. Obviously, a free oscillation of positively damped systems with algebraically decaying amplitudes can be expressed by the set of base functions

$$\{(1+\lambda t)^{-m}\sin(n\omega t), (1+\lambda t)^{-m} \times \cos(n\omega t) \mid m \geqslant 1,\ n \geqslant 0\}, \tag{3}$$

where the two unknown time-constants ω and λ relate to the two time scales

$$\tau_1 = \omega t, \qquad \tau_2 = \lambda t, \tag{4}$$

respectively. Then, one has $u(t) = u(\tau_1, \tau_2)$ and

$$\dot{u} = \omega \frac{\partial u}{\partial \tau_1} + \lambda \frac{\partial u}{\partial \tau_2},$$

$$\ddot{u} = \omega^2 \frac{\partial^2 u}{\partial \tau_1^2} + 2\omega\lambda \frac{\partial^2 u}{\partial \tau_1 \partial \tau_2} + \lambda^2 \frac{\partial^2 u}{\partial \tau_2^2}. \tag{5}$$

Thus, the original governing equation (1) becomes

$$\omega^2 \frac{\partial^2 u}{\partial \tau_1^2} + 2\omega\lambda \frac{\partial^2 u}{\partial \tau_1 \partial \tau_2} + \lambda^2 \frac{\partial^2 u}{\partial \tau_2^2} = f\left[\omega \frac{\partial u}{\partial \tau_1} + \lambda \frac{\partial u}{\partial \tau_2}, u\right] \tag{6}$$

subject to the corresponding initial conditions

$$u(\tau_1, \tau_2) = \alpha, \qquad \omega \frac{\partial u(\tau_1, \tau_2)}{\partial \tau_1} + \lambda \frac{\partial u(\tau_1, \tau_2)}{\partial \tau_2} = \beta, \quad \text{when } \tau_1 = \tau_2 = 0. \tag{7}$$

Notice that it is *unnecessary* for us to assume the existence of any small parameters in (1) and (2) at all.

2. Basic ideas

It is a fact that a real function $f(x)$ can be represented by different sets of base functions. Besides, a real function can be *more efficiently* represented by a *better* set of base function. So, it is important to approximate a given non-linear problem by a proper set of base functions. Practically, this provides us with the beginning point to apply the homotopy analysis method. Obviously, according to (3) and the definition (4), the considered free oscillations can be represented by the set of base functions

$$\{(1+\tau_2)^{-m}\sin(n\tau_1),(1+\tau_2)^{-m}\times\cos(n\tau_1)\,|\,m\geqslant 1,\ n\geqslant 0\} \tag{8}$$

such that

$$u(\tau_1,\tau_2)=\sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty}(1+\tau_2)^{-m}\times[a_{m,n}\sin(n\tau_1)+b_{m,n}\cos(n\tau_1)], \tag{9}$$

where $a_{m,n}$, $b_{m,n}$ are coefficients. This provides us with a rule, called *the Rule of Solution Expression* for the sake of brevity.

The homotopy analysis method is based on continuous variations from initial guesses to the exact solution of a considered problem. Mathematically speaking, for the problem under consideration, one constructs the continuous mappings $u(\tau_1,\tau_2)\to\Phi(\tau_1,\tau_2,p)$, $\omega\to\Omega(p)$ and $\lambda\to\Lambda(p)$ in such a way that $\Phi(\tau_1,\tau_2,p)$, $\Omega(p)$ and $\Lambda(p)$ vary from their initial guesses $u_0(\tau_1,\tau_2),\omega_0,\lambda_0$ to their exact solutions $u(\tau_1,\tau_2),\omega,\lambda$, respectively, as p increases from 0 to 1. To ensure this, one constructs a family (in the embedding parameter p) of non-linear differential equations

$$(1-p)\{\mathscr{L}[\Phi(\tau_1,\tau_2,p)]-\mathscr{L}_0[u_0(\tau_1,\tau_2)]\}=\hbar\,p\mathscr{N}[\Phi(\tau_1,\tau_2,p)] \tag{10}$$

subject to the related initial conditions

$$\Phi(\tau_1,\tau_2,p)=\alpha,\quad \tau_1=\tau_2=0 \tag{11}$$

and

$$\omega\frac{\partial\Phi(\tau_1,\tau_2,p)}{\partial\tau_1}+\lambda\frac{\partial\Phi(\tau_1,\tau_2,p)}{\partial\tau_2}=\beta,\quad \tau_1=\tau_2=0, \tag{12}$$

where $u_0(\tau_1,\tau_2)$ is an initial guess of $u(\tau_1,\tau_2)$, $\hbar$ is an auxiliary non-zero parameter, $\mathscr{N}$ is a non-linear operator defined by

$$\begin{aligned}\mathscr{N}[\Phi(\tau_1,\tau_2,p)]=\ &\Omega^2(p)\frac{\partial^2\Phi(\tau_1,\tau_2,p)}{\partial\tau_1^2}\\&+2\Omega(p)\Lambda(p)\frac{\partial^2\Phi(\tau_1,\tau_2,p)}{\partial\tau_1\partial\tau_2}\\&+\Lambda^2(p)\frac{\partial^2\Phi(\tau_1,\tau_2,p)}{\partial\tau_2^2}\\&-f\left[\Omega(p)\frac{\partial\Phi(\tau_1,\tau_2,p)}{\partial\tau_1}\right.\\&\left.+\Lambda(p)\frac{\partial\Phi(\tau_1,\tau_2,p)}{\partial\tau_2},\Phi(\tau_1,\tau_2,p)\right]\end{aligned} \tag{13}$$

and $\mathscr{L}$ and $\mathscr{L}_0$ are two auxiliary linear operators defined by

$$\begin{aligned}&\mathscr{L}[\Phi(\tau_1,\tau_2,p)]\\&=\Omega^2(p)\frac{\partial^2\Phi}{\partial\tau_1^2}+2\Omega(p)\Lambda(p)(1+\tau_2)\frac{\partial^2\Phi}{\partial\tau_1\partial\tau_2}\\&\quad+\Lambda(p)^2(1+\tau_2)^2\frac{\partial^2\Phi}{\partial\tau_2^2}+\mu_1\left[\Omega(p)\frac{\partial\Phi}{\partial\tau_1}\right.\\&\quad\left.+\Lambda(p)(1+\tau_2)\frac{\partial\Phi}{\partial\tau_2}\right]+\mu_2\Phi\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mathscr{L}_0[u_0(\tau_1,\tau_2)]=&\omega_0^2\frac{\partial^2u_0}{\partial\tau_1^2}+2\omega_0\lambda_0(1+\tau_2)\frac{\partial^2u_0}{\partial\tau_1\partial\tau_2}\\&+\lambda_0^2(1+\tau_2)^2\frac{\partial^2u_0}{\partial\tau_2^2}+\mu_1\left[\omega_0\frac{\partial u_0}{\partial\tau_1}\right.\\&\left.+\lambda_0(1+\tau_2)\frac{\partial u_0}{\partial\tau_2}\right]+\mu_2u_0,\end{aligned} \tag{15}$$

respectively, in which ω_0, λ_0 are initial guesses of the time-constants ω and λ, and μ_1, μ_2 are two constants to be determined later. Notice that $\Omega(p)$ and $\Lambda(p)$ in (13) and (14) are related with the time-constants ω and λ, respectively.

Under *the Rule of Solution Expression*, it is natural for us to choose

$$u_0(\tau_1,\tau_2) = a_0\cos\tau_1 + (a_1\cos\tau_1 + b_1\sin\tau_1) \times(1+\tau_2)^{-1}, \tag{16}$$

as the initial guess of $u(\tau_1,\tau_2)$, where a_0, a_1 and b_1 are unknown constants to be determined later.

When $p=0$, Eq. (10) has the solution

$$\Omega(0)=\omega_0, \quad \Lambda(0)=\lambda_0,$$
$$\Phi(\tau_1,\tau_2,0)=u_0(\tau_1,\tau_2) \tag{17}$$

and the corresponding initial conditions become

$$u_0(\tau_1,\tau_2)=\alpha, \qquad \omega\frac{\partial u_0(\tau_1,\tau_2)}{\partial\tau_1}+\lambda\frac{\partial u_0(\tau_1,\tau_2)}{\partial\tau_2}=\beta,$$
$$\tau_1=\tau_2=0. \tag{18}$$

Substituting (16) into the above expression and setting $\omega=\omega_0$, $\lambda=\lambda_0$, one has

$$a_1=\alpha-a_0, \quad b_1=\frac{\alpha\lambda_0+\beta}{\omega_0}. \tag{19}$$

Notice that we still have one more degree of freedom here, i.e. the value of a_0. Because the amplitude of the oscillation algebraically decays to zero, all pure periodic (trigonometric) terms should decay to zero as $u(\tau_1,\tau_2)$ becomes more and more accurate. This provides us with a simple method to check whether or not the proposed method works. In practice, one often sets a_0 a small value. Thus, as long as a_0 and the initial guesses ω_0, λ_0 are given, the initial guess $u_0(\tau_1,\tau_2)$ is determined.

When $p=1$, Eqs. (10)–(12) are exactly the same as (6) and (7), provided

$$\Omega(1)=\omega, \quad \Lambda(1)=\lambda. \tag{20}$$

Thus, it holds

$$\Phi(\tau_1,\tau_2,1)=u(\tau_1,\tau_2). \tag{21}$$

Therefore, as the embedding parameter p increases from 0 to 1, $\Phi(\tau_1,\tau_2,p)$ indeed varies from the initial trial $u_0(\tau_1,\tau_2)$ to the exact solution $u(\tau_1,\tau_2)$ of the original Eqs. (6) and (7); so do $\Omega(p)$ and $\Lambda(p)$ as they vary from initial guesses ω_0,λ_0 to the time-constants ω and λ which relate to the physical time scales $\tau_1=\omega t$ and $\tau_2=\lambda t$, respectively. This kind of variations is called deformation in topology. For brevity, Eqs. (10)–(12) are called the zero-order deformation equations.

By Taylor's theorem and due to (17), $\Omega(p)$, $\Lambda(p)$ and $\Phi(\tau_1,\tau_2,p)$ can be expanded in power series of p as follows:

$$\Omega(p)\sim\omega_0+\sum_{k=1}^{+\infty}\frac{\omega_0^{[k]}}{k!}\,p^k, \tag{22}$$

$$\Lambda(p)\sim\lambda_0+\sum_{k=1}^{+\infty}\frac{\lambda_0^{[k]}}{k!}\,p^k, \tag{23}$$

$$\Phi(\tau_1,\tau_2,p)\sim u_0(\tau_1,\tau_2)+\sum_{k=0}^{+\infty}\frac{u_0^{[k]}(\tau_1,\tau_2)}{k!}\,p^k, \tag{24}$$

where

$$\omega_0^{[k]}=\left.\frac{d^k\Omega(p)}{dp^k}\right|_{p=0}, \quad \lambda_0^{[k]}=\left.\frac{d^k\Lambda(p)}{dp^k}\right|_{p=0},$$
$$u_0^{[k]}(\tau_1,\tau_2)=\left.\frac{\partial^k\Phi(\tau_1,\tau_2,p)}{\partial p^k}\right|_{p=0} \tag{25}$$

are called kth-order deformation derivatives. Notice that there exists an auxiliary non-zero parameter $\hbar$ on the right-hand side of (10), which influences the convergence regions of the series (22)–(24). This provides us with a convenient way to *adjust* and *control* the convergence of approximations. Assume that the value of $\hbar$ is so properly chosen that these series converge at $p=1$. Liao [24] proved in rather general cases that a convergent series given by the homotopy analysis method (at $p=1$) must be an exact solution of a considered non-linear problem. Thus, one has

$$\omega=\omega_0+\sum_{k=1}^{+\infty}\omega_k, \tag{26}$$

$$\lambda=\lambda_0+\sum_{k=1}^{+\infty}\lambda_k, \tag{27}$$

$$u(\tau_1,\tau_2)=u_0(\tau_1,\tau_2)+\sum_{k=0}^{+\infty}u_k(\tau_1,\tau_2). \tag{28}$$

where

$$\omega_k = \frac{\omega_0^{[k]}}{k!}, \quad \lambda_k = \frac{\lambda_0^{[k]}}{k!},$$

$$u_k(\tau_1,\tau_2) = \frac{u_0^{[k]}(\tau_1,\tau_2)}{k!}. \tag{29}$$

The solutions to the mth-order approximation are

$$u(t) \approx \sum_{k=0}^{m} u_k(\tau_1,\tau_2), \tag{30}$$

$$\omega \approx \sum_{k=0}^{m} \omega_k, \tag{31}$$

$$\lambda \approx \sum_{k=0}^{m} \lambda_k, \tag{32}$$

where $\tau_1 = \omega t$ and $\tau_2 = \lambda t$. The governing equation and initial conditions of $u_k(\tau_1,\tau_2)$ $(k = 1,2,3,\ldots)$ can be deduced from the zero-order deformation Eqs. (10)–(12). First differentiating Eqs. (10)–(12) k times with respect to p and then setting $p = 0$ and finally dividing them by $k!$, one obtains the so-called kth-order deformation equation

$$\mathscr{L}_0[u_k(\tau_1,\tau_2) - \chi_k u_{k-1}(\tau_1,\tau_2)] = \hbar R_k(\tau_1,\tau_2) - W_k(\tau_1,\tau_2) + \chi_k W_{k-1}(\tau_1,\tau_2), \tag{33}$$

subject to the initial conditions (when $\tau_1 = \tau_2 = 0$)

$$u_k(\tau_1,\tau_2) = 0 \tag{34}$$

and

$$\omega \frac{\partial u_k}{\partial \tau_1} + \lambda \frac{\partial u_k}{\partial \tau_2} = 0, \tag{35}$$

where

$$W_k(\tau_1,\tau_2) = \sum_{n=1}^{k} \left[\left(\sum_{j=0}^{n} \omega_j \omega_{n-j} \right) \frac{\partial^2 u_{k-n}}{\partial \tau_1^2} + 2 \left(\sum_{j=0}^{n} \omega_j \lambda_{n-j} \right) (1+\tau_2) \frac{\partial^2 u_{k-n}}{\partial \tau_1 \partial \tau_2} + \left(\sum_{j=0}^{n} \lambda_j \lambda_{n-j} \right) (1+\tau_2)^2 \frac{\partial^2 u_{k-n}}{\partial \tau_2^2} + \mu_1 \left(\omega_n \frac{\partial u_{k-n}}{\partial \tau_1} + \lambda_n (1+\tau_2) \frac{\partial u_{k-n}}{\partial \tau_2} \right) \right] \tag{36}$$

and

$$R_k(\tau_1,\tau_2) = \frac{1}{(k-1)!} \left. \frac{d^{k-1} \mathscr{N}[\Phi(\tau_1,\tau_2,p)]}{d\, p^{k-1}} \right|_{p=0}, \tag{37}$$

in which $\mathscr{N}[\Phi(\tau_1,\tau_2,p)]$ is given by (13) and χ_k is defined by

$$\chi_k = \begin{cases} 0, & k \leqslant 1, \\ 1, & k > 1. \end{cases} \tag{38}$$

In view of (35), the initial condition becomes

$$\left(\omega \frac{\partial}{\partial \tau_1} + \lambda \frac{\partial}{\partial \tau_2} \right) \left(\sum_{j=0}^{k} u_j(\tau_1,\tau_2) \right) = \beta, \tag{39}$$

where the two unknowns ω and λ are replaced by their kth-order approximations defined by (31) and (32), respectively.

Under *the Rule of Solution Expression*, the values of μ_1 and μ_2 are determined in such a way that for any constants B_1 and B_2 the equation

$$\mathscr{L}_0[(1+\tau_2)^{-1}(B_1 \sin\tau_1 + B_2 \cos\tau_1)] = 0 \tag{40}$$

holds. Due to (15), the above equation becomes

$$(\mu_1 - 2\lambda_0)\omega_0 \frac{(B_1 \cos\tau_1 - B_2 \sin\tau_1)}{1+\tau_2} + (\mu_2 - \lambda_0\mu_1 + 2\lambda_0^2 - \omega_0^2) \times \frac{(B_1 \sin\tau_1 + B_2 \cos\tau_1)}{1+\tau_2} = 0,$$

which holds for any B_1 and B_2, provided

$$\mu_1 - 2\lambda_0 = 0, \qquad \mu_2 - \lambda_0\mu_1 + 2\lambda_0^2 - \omega_0^2 = 0.$$

Solving above set of linear algebraic equations, one has

$$\mu_1 = 2\lambda_0, \qquad \mu_2 = \omega_0^2. \tag{41}$$

Thus, like the initial guess $u_0(\tau_1,\tau_2)$, the auxiliary linear operator $\mathscr{L}$ in general and $\mathscr{L}_0$ in particular (at $p=0$) are uniquely determined by the initial guesses ω_0,λ_0 under *the Rule of Solution Expression*.

Notice that $W_k(\tau_1,\tau_2)$ $(k=1,2,3,\ldots)$ contains the terms $(1+\tau_2)^{-1}\sin(\tau_1)$ and $(1+\tau_2)^{-1}\cos(\tau_1)$, and their coefficients contain the unknowns ω_k and λ_k. Due to (40), if $(1+\tau_2)^{-1}\sin(\tau_1)$ and/or $(1+\tau_2)^{-1}\cos(\tau_1)$ appear on the right-hand side of (33), its solution contains the so-called secular terms $\tau_1\sin\tau_1/(1+\tau_2)$ and $\tau_1\cos\tau_1/(1+\tau_2)$, which however disobey *the Rule of Solution Expression*, more clearly described by (9). To avoid this, one had to set the coefficients of these two terms to zero. This gives us a set of two linear algebraic equations for ω_k and λ_k. In this way ω_k and λ_k are uniquely determined.

To gain uniformly valid approximations, some perturbation techniques were developed to avoid the appearance of so-called secular terms in perturbation solutions. This kind of technique goes back to some scientists in the 19th century, such as Lindstedt [28], Bohlin [29], Poincaré [30], Gyldén [31] and so on. The idea was further developed by Lighthill [32,33], Malkin [34], Kuo [35,36], Tsien [37] and so on. *The Rule of Solution Expression* can be seen as a kind of generalization of this idea.

Then, the right-hand side of (33) only contains the terms $(1+\tau_2)^{-m}\sin(n\tau_1)$ and $(1+\tau_2)^{-m}\cos(n\tau_1)$, where $(m-1)^2+(n-1)^2>0$. It is very easy to solve such a linear equation. For simplicity, define an operator $\mathscr{L}_0^{\mathrm{inv}}$ such that

$$\mathscr{L}_0^{\mathrm{inv}}[(1+\tau_2)^{-m}\sin(n\tau_1)] = (1+\tau_2)^{-m}[\sigma_1\sin(n\tau_1)+\sigma_2\cos(n\tau_1)] \tag{42}$$

and

$$\mathscr{L}_0^{\mathrm{inv}}[(1+\tau_2)^{-m}\cos(n\tau_1)] = (1+\tau_2)^{-m}[\sigma_3\cos(n\tau_1)+\sigma_4\sin(n\tau_1)], \tag{43}$$

where $(m-1)^2+(n-1)^2>0$. Enforcing $\mathscr{L}_0\mathscr{L}_0^{\mathrm{inv}}=\mathscr{L}_0^{\mathrm{inv}}\mathscr{L}_0=1$, one has

$$\sigma_1 = \frac{m(m-1)\lambda_0^2-(n^2-1)\omega_0^2}{[m(m-1)\lambda_0^2-(n^2-1)\omega_0^2]^2+4(m-1)^2n^2(\omega_0\lambda_0)^2}, \tag{44}$$

$$\sigma_2 = \frac{2(m-1)n\omega_0\lambda_0}{[m(m-1)\lambda_0^2-(n^2-1)\omega_0^2]^2+4(m-1)^2n^2(\omega_0\lambda_0)^2} \tag{45}$$

and $\sigma_3=\sigma_1$, $\sigma_4=-\sigma_2$. Therefore,

$$u_k^*(\tau_1,\tau_2)=\chi_k u_{k-1}(\tau_1,\tau_2)+\mathscr{L}_0^{\mathrm{inv}}\{\hbar R_k(\tau_1,\tau_2) - W_k(\tau_1,\tau_2)+\chi_k W_{k-1}(\tau_1,\tau_2)\} \tag{46}$$

is a special solution of (33), which can be obtained *merely* by means of some algebraic calculations. The related common solution is

$$u_k(\tau_1,\tau_2)=u_k^*(\tau_1,\tau_2)+(1+\tau_2)^{-1}\times(C_1\sin\tau_1+C_2\cos\tau_1). \tag{47}$$

The coefficients C_1 and C_2 can be determined by the two initial conditions (34) and (39). Notice that Eq. (33) might have solutions in many other forms, which, however, disobey expression (9) and therefore must be eliminated under *the Rule of Solution Expression*. Therefore, the above solution is the most general form of $u_k(\tau_1,\tau_2)$ $(k=1,2,3,\ldots)$ under *the Rule of Solution Expression*.

Finally, the whole process of the proposed method is briefly described. First, set a_0 a small value and properly select initial guesses ω_0 and λ_0. They uniquely determine the initial guess $u_0(\tau_1,\tau_2)$ by (16) and (19), and the auxiliary linear operator $\mathscr{L}_0$ by (15) and (41), respectively. Then, enforcing the coefficients of $(1+\tau_2)^{-1}\sin\tau_1$ and $(1+\tau_2)^{-1}\cos\tau_1$ on the right-hand side of the 1st-order $(k=1)$ deformation equation (33) to be zero, one gains a set of linear algebraic equations for ω_1 and λ_1, which determines ω_1 and λ_1. Then, all coefficients in (33) and (39) are known, and a special solution of the 1st-order deformation equation (33) can be easily obtained by means of the operator $\mathscr{L}_0^{\mathrm{inv}}$ defined by (42) and (43). The integral coefficients C_1 and C_2 in the common solution (47) are determined by the two initial conditions (34) and (39). Similarly, one can further calculate ω_2, λ_2, $u_2(\tau_1,\tau_2)$ and so on.

3. Some examples

In this section, the validity, effectiveness and flexibility of the proposed approach is illustrated by some

examples. The symbolic calculation software Mathematica is employed.

Unlike *all* other analytic techniques for non-linear problems, the homotopy analysis method gives a one-parameter *family* (in the auxiliary parameter $\hbar$) of results at *any* given order of approximations. It is the auxiliary parameter $\hbar$ which provides us with a convenient way to *adjust* and *control* the convergence of approximations.

Besides, for rather general cases Liao [24] proved that any convergent series given by the homotopy analysis method at $p=1$ must be one of the exact solutions of a considered non-linear problem. So, given initial trials ω_0, λ_0 and the auxiliary parameter a_0, one only needs to choose a proper value for $\hbar$ to ensure that series (26)–(28) converge.

3.1. Example 1

Consider [4, p. 112]

$$\ddot{u}+u+\varepsilon\dot{u}^3=0 \tag{48}$$

subject to the initial conditions (2).

All related formulae are the same as those mentioned in Section 2, except

$$R_k(\tau_1,\tau_2)=w_{k-1}+u_{k-1}+\varepsilon\sum_{n=0}^{k-1}\left(\sum_{j=0}^{n}v_jv_{n-j}\right)v_{k-1-n}, \tag{49}$$

where

$$v_n=\sum_{j=0}^{n}\left(\omega_j\frac{\partial u_{n-j}}{\partial\tau_1}+\lambda_j\frac{\partial u_{n-j}}{\partial\tau_2}\right) \tag{50}$$

and

$$w_n=\sum_{j=0}^{n}\left\{\left(\sum_{i=0}^{j}\omega_i\omega_{j-i}\right)\frac{\partial^2u_{n-j}}{\partial\tau_1^2}+2\left(\sum_{i=0}^{j}\omega_i\lambda_{j-i}\right)\frac{\partial^2u_{n-j}}{\partial\tau_1\partial\tau_2}+\left(\sum_{i=0}^{j}\lambda_i\lambda_{j-i}\right)\frac{\partial^2u_{n-j}}{\partial\tau_2^2}\right\}. \tag{51}$$

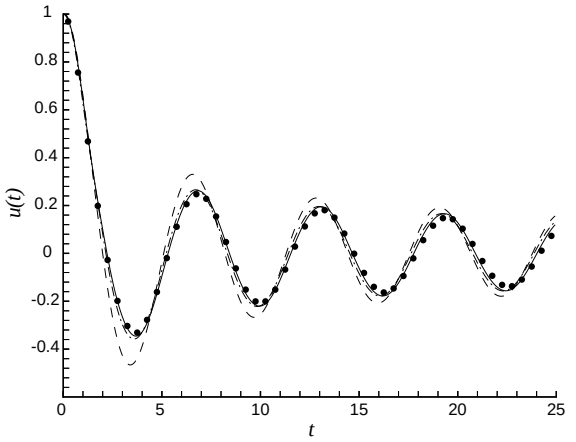

Fig. 1. Comparison of the exact solution with the analytic approximations of the 1st example when $\alpha=1$, $\beta=0$, $\varepsilon=3$ by means of $\hbar=-1$, $a_0=1/10$, $\lambda_0=1/2$ and $\omega_0=1$. Symbols: numerical solution; dashed line: zero-order approximation; dash–dotted line: 3rd-order approximation; solid line: 6th-order approximation.

We calculated a large number of cases for different physical parameters and initial conditions. The calculations indicate that the proposed approach is indeed valid. To show this, two cases are illustrated. One is $\alpha=1$, $\beta=0$, $\varepsilon=3$. The other is $\alpha=1$, $\beta=0$, $\varepsilon=5$. In the former case, even the 6th-order approximation, in which $\hbar=-1$, $\omega_0=1$, $\lambda_0=1/2$ and $a_0=1/10$, agrees well with the exact result, as shown in Fig. 1. In the latter case, the 12th-order approximation with $\hbar=-1/2$, $\omega_0=1$, $\lambda_0=1/2$ and $a_0=1/20$ agrees well with the exact result, as shown in Fig. 2. Generally, for problems with stronger non-linearity, the absolute value of $\hbar$ ($\hbar<0$) had to be decreased so as to *adjust* and *control* the convergence of approximations. For example, $\hbar$ in the latter case is closer to 0 than that in the former, because the non-linearity of the latter is stronger than the former. This is a general principle of the choice of $\hbar$, as reported in our previous publications [21,22,24,26,13].

3.2. Example 2

Consider, next [4, p. 121]

$$\ddot{u}+u=\varepsilon u\dot{u}\ddot{u}, \tag{52}$$

again subject to the initial conditions (2).

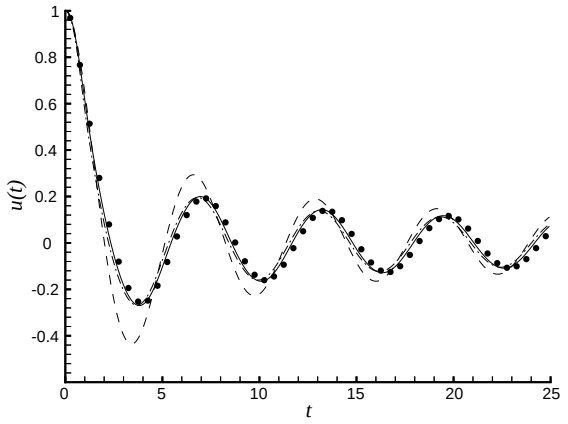

Fig. 2. Comparison of the exact solution with the analytic approximations of the first example when $\alpha = 1$, $\beta = 0$, $\varepsilon = 5$ by means of $\hbar = -1/2$, $a_0 = 1/20$, $\lambda_0 = 1/2$ and $\omega_0 = 1$. Symbols: numerical solution; dashed line: zero-order approximation; dash–dotted line: 6th-order approximation; solid line: 12th-order approximation.

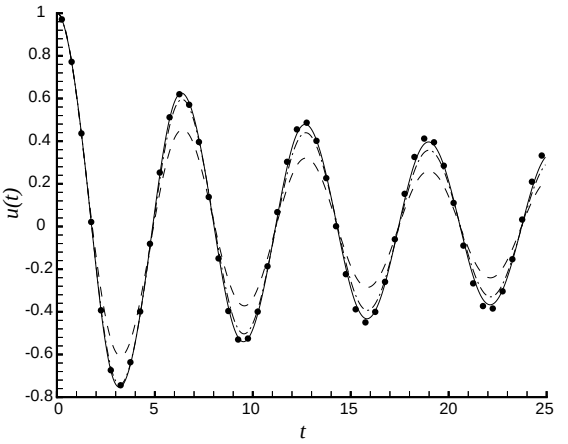

Fig. 3. Comparison of the exact solution with the analytic approximations of the 2nd example when $\alpha = 1$, $\beta = 0$, $\varepsilon = 1$ by means of $\hbar = -3/4$, $a_0 = 1/10$, $\lambda_0 = 1/4$ and $\omega_0 = 1$. Symbols: numerical solution, dashed line: zero-order approximation, dash–dotted line: 2nd-order approximation, solid line: 4th-order approximation.

All formulae are the same as those mentioned in Section 2, except

$$R_k(\tau_1, \tau_2) = w_{k-1} + u_{k-1} - \varepsilon \sum_{n=0}^{k-1} u_{k-1-n} \left(\sum_{j=0}^{n} v_j w_{n-j} \right), \qquad (53)$$

where w_j and v_j are defined by (51) and (50), respectively.

Similarly, a large number of cases for different physical parameters and initial conditions are calculated, which verify the validity of the proposed approach. Two cases are employed to illustrate this. One is $\alpha = 1$, $\beta = 0$, $\varepsilon = 1$. The other is $\alpha = 1$, $\beta = 0$, $\varepsilon = 5$. In the former case, even the 4th-order approximation, in which $\hbar = -3/4$, $\omega_0 = 1$, $\lambda_0 = 1/4$ and $a_0 = 1/10$, agrees well with the exact result, as shown in Fig. 3. In the latter case, our 5th-order approximation with $\hbar = -1/2$, $\omega_0 = 1$, $\lambda_0 = 1/4$ and $a_0 = 1/10$ agrees well with the exact result, as shown in Fig. 4. Notice that the non-linearity of the latter case is stronger than that of the former. So, to *control* the convergence of approximations, the value of $\hbar$ in the latter case is *chosen* to be closer to 0 than that in the former case. Besides, to obtain sufficiently accurate enough results, the order of approximation in the latter case is higher than the former. This is similar to

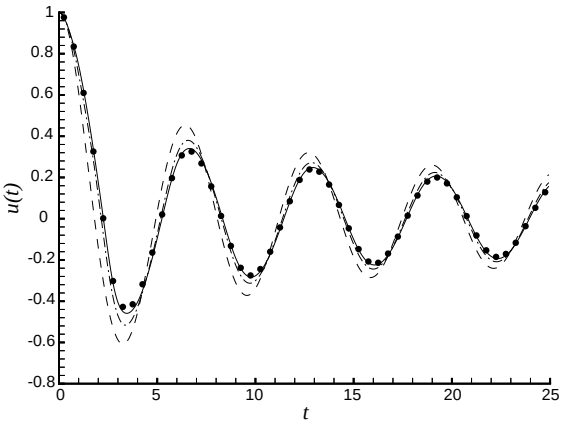

Fig. 4. Comparison of the exact solution with the analytic approximations of the 2nd example when $\alpha = 1$, $\beta = 0$, $\varepsilon = 5$ by means of $\hbar = -1/2$, $a_0 = 1/10$, $\lambda_0 = 1/4$ and $\omega_0 = 1$. Symbols: numerical solution, dashed line: zero-order approximation, dash–dotted line: 2nd-order approximation, solid line: 5th-order approximation.

the first example and has been reported in our previous publications.

Notice that one has great freedom to choose the initial trials ω_0 and λ_0, and the auxiliary parameter a_0. Our calculations indicate that the validity of the proposed approach is not sensitive to the choice of ω_0, λ_0 and a_0. For example, we calculated the latter case

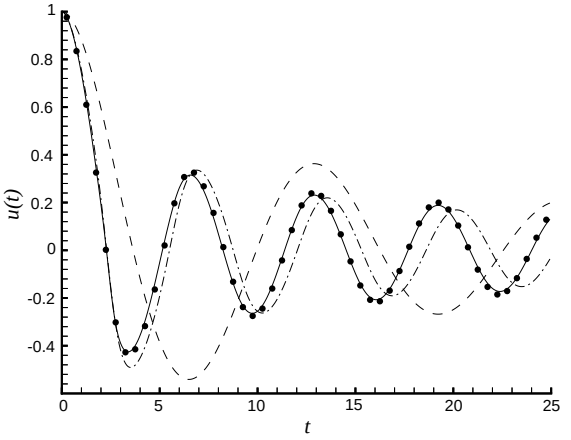

Fig. 5. Comparison of the exact solution with the analytic approximations of the 2nd example when $\alpha=1$, $\beta=0$, $\varepsilon=5$ by means of $\hbar=-1/4$, $a_0=-1/20$, $\lambda_0=1/8$ and $\omega_0=1/2$. Symbols: numerical solution, dashed line: zero-order approximation, dash–dotted line: 6th-order approximation, solid line: 12th-order approximation.

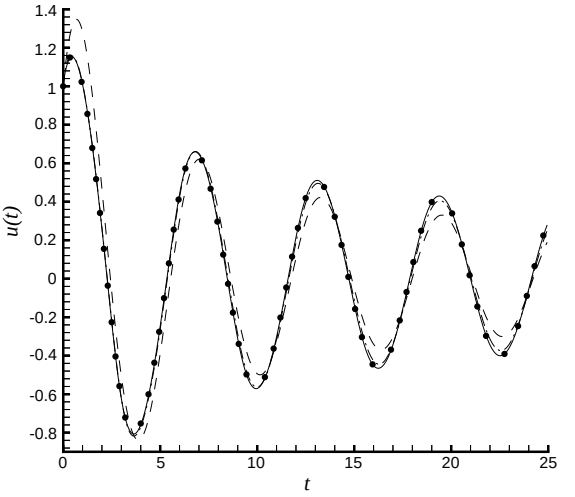

Fig. 6. The analytic approximations of the 2nd example when $\alpha=1$, $\beta=1$, $\varepsilon=1$ by means of $\hbar=-3/4$, $a_0=1/10$, $\lambda_0=1/4$ and $\omega_0=1$. Dashed line: zero-order approximation; dash–dotted line: 4th-order approximation; solid line: 10th-order approximation; symbols: 12th-order approximation.

of the second example by means of $\omega_0=1/2$, $\lambda_0=1/8$, $a_0=-1/20$ and $\hbar=-1/4$. Although the initial guess is much worse, our 12th-order approximation agrees well with the exact solution, as shown in Fig. 5.

To employ Runge–Kutta's method to get numerical result, one rewrites Eq. (52) as

$$\ddot{u}=-\frac{u}{(1-\varepsilon u\dot{u})}.$$

If $1-\varepsilon u\dot{u}=0$ at any time $t>0$, the Runge–Kutta's method fails. This is the case when $\alpha=1$, $\beta=1$ and $\varepsilon=1$. However, the proposed analytic approach can still give convergent result when a_0, λ_0, ω_0 and $\hbar$ are properly chosen (we choose here $a_0=1/10$, $\lambda_0=1/4$, $\omega_0=1$ and $\hbar=-3/4$), as shown in Fig. 6. Note that even the 4th-order approximation is rather close to the 12th-order approximation, and the 10th-order approximation is nearly the same as the 12th-order one. Thus, it is clear that the series of the analytic approximations converges. Then, due to Liao's [24] convergence theorem, it is quite sure that the convergent series must be the exact solution of the 2nd example when $\alpha=1$, $\beta=1$ and $\varepsilon=1$, even if the corresponding numerical result can*not* be given by Runge–Kutta's method. This illustrates the validity and potential of the homotopy analysis method once again.

4. Conclusions and discussions

In this paper, the homotopy analysis method [20–27,13] is employed to propose a new approach for free oscillations of positively damped systems with algebraically decaying amplitude. In contrast to perturbation techniques, this approach is valid even for damped systems without any small/large parameters. Thus, it can be applied more widely than perturbation techniques, especially when there do not exist any small/large parameters. Two examples are employed to illustrate its validity, effectiveness and flexibility.

In the frame of the homotopy analysis method the proposed approach provides us with a convenient way to *adjust* and *control* convergence of approximations by choosing a proper value for the auxiliary parameter $\hbar$. Besides, one fundamental rule, i.e. *the Rule of Solution Expression*, is proposed, and its importance on choosing initial guesses and the auxiliary linear operator is shown. Notice that the auxiliary linear operator $\mathscr{L}$ can now be defined as a function of the embedding parameter p. All of these indicate the validity and flexibility of the homotopy analysis method.

In our previous applications of the homotopy analysis method, we focused on either steady or periodic

problems. In this paper, the homotopy analysis method is further improved to be valid for unsteady problems. The success of the proposed approach indicates that the homotopy analysis method is indeed valid for unsteady non-linear problems. Besides, it should be emphasized that the homotopy analysis method is valid even if corresponding numerical results cannot be given by Runge–Kutta's method, as illustrated in this paper. All of these verify once again that the homotopy analysis method is indeed a promising analytic tool for non-linear problems in science and engineering.

Acknowledgements

Thanks to professor Peter Hagedorn (Darmstadt University of Technology) and the anonymous reviewers for their helpful suggestions and improving the wording. This work is partly supported by National Science Fund for Distinguished Young Scholars of China (Approval no. 50125923).

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