J. Non-Newtonian Fluid Mech. 129 (2005) 46–55

Journal of Non-Newtonian Fluid Mechanics

www.elsevier.com/locate/jnnfm

Series solutions of unsteady magnetohydrodynamic flows of non-Newtonian fluids caused by an impulsively stretching plate

Hang Xu, Shi-Jun Liao*

School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200030, China

Received 27 November 2004; received in revised form 7 April 2005; accepted 8 May 2005

Abstract

In this paper, the unsteady magnetohydrodynamic viscous flows of non-Newtonian fluids caused by an impulsively stretching plate are studied by means of an analytic technique, namely the homotopy analysis method. We give the analytic series solutions which are accurate and uniformly valid for all dimensionless time in the whole spatial region $0 \le \eta < \infty$. To the best of authors' knowledge, such kind of analytic solutions have been never reported. Besides, the effects of the integral power-law index ($n = 1, 2, 3$) of the non-Newtonian fluids and the magnetic parameter $M = 0, 1, 2$ on the flows are investigated.

Keywords: Unsteady; non-Newtonian fluid; power-law index; MHD; Series solution

1. Introduction

Investigations of boundary layer flows of viscous fluids due to a stretching sheet have been the interest of many researchers owing to its important applications in chemical and metallurgical industries, such as polymer extrusion, drawing of copper wires, continuous stretching of plastic films and artificial fibers, hot rolling, wire drawing, glass-fiber, metal extrusion, and metal spinning. Sakiadis [1] initiated the study of the boundary layer flow over a stretched surface moving with a constant velocity and formulated a boundary-layer equation for two-dimensional and axisymmetric flows. Tsou et al. [2] considered the effect of heat transfer in the boundary on a continuous moving surface with a constant velocity and experimentally confirmed the numerical results of Sakiadis. Erickson et al. [3] extended the work of Sakiadis to include blowing or suction at the stretched sheet surface on a continuous solid surface under constant speed and investigated its effects on the heat and mass transfer in the boundary layer. Chen and Stroble [4] investigated the effect of a buoyancy-induced pressure gradient in a laminar boundary layer of a stretched sheet with constant surface velocity and temperature. Jacobi [5] reported numerical results for a stretched surface with uniform motion. The related problems of a stretched sheet with a linear velocity and different thermal boundary conditions are studied, theoretically, numerically and experimentally, by many researchers such as Crane [6], Chen and Char [7], Gupta and Gupta [8]. Rajagopal [9] studied a boundary layer flow of a non-Newtonian fluid due to a stretching sheet with uniform free stream, and obtained many interesting results. Troy et al. [10] established the uniqueness of the steady flow of an incompressible second-order fluid over a stretching sheet.

In recent years, many investigations have concentrated on the magnetohydrodynamic (MHD) flows because of its important applications in metallurgical industry, such as the cooling of continuous strips and filaments drawn through a quiescent fluid and the purification of molten metals from non-metallic inclusions. Chakrabarti and Gupta [11] studied the MHD flow of Newtonian fluids initially at rest, over a stretching sheet at a different uniform temperature. Vajravelu and Hadjinicolaou [12] made a analysis to flows and heat transfer characteristics in an electrically conducting fluid near an isothermal sheet. Many works have been reported on the problem of a stretching sheet with a linear velocity in the presence of a magnetic fluid, such as [13–18]. The MHD

* Corresponding author. Tel.: +86 21 6293 2676; fax: +86 21 6293 3156.
E-mail addresses: henry629@sjtu.edu.cn (H. Xu), sjliao@sjtu.edu.cn (S.-J. Liao).

0377-0257/$ – see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.jnnfm.2005.05.005

flows of non-Newtonian fluids were initially studied by Sarpkaya [19], then followed by Djukic [20,21] , Andersson et al. [22], and Liao [23], etc. Very recently, Liao [24] obtained an accurate analytic solution of unsteady boundary layer flows caused by an impulsively stretching plate uniformly valid for all non-dimensional time, τ. Cheng and Huang [25] considered the problem of unsteady flows and heat transfer in the laminar boundary on a linearly accelerating surface with suction or blowing in the absence and presence of a heat source or sink.

The homotopy [26] is a basic concept in topology [27]. Based on the homotopy, some numerical techniques such as the continuation method [28] and the homotopy continuation method [29] were developed. There is a suite of FORTRAN subroutines in Netlib for solving nonlinear systems of equations by homotopy methods, called HOMPACK. Currently, using the concept of homotopy, Liao [30] developed a new analytic method for highly nonlinear problems, namely the homotopy analysis method (HAM). Different from perturbation techniques [31], the homotopy analysis method does not depend upon any small or large parameters and thus is valid for most of nonlinear problems in science and engineering. Besides, it logically contains other non-perturbation techniques such as Lyapunov's small parameter method [32], the δ-expansion method [33], and Adomian's decomposition method [34]. The homotopy analysis method has been successfully applied to many nonlinear problems [35–40]. The aim of the present paper is to study the unsteady MHD viscous flows of non-Newtonian fluids caused by an impulsively stretching plate, and to investigate the effect of integral power-law index of these non-Newtonian fluids on the velocity. To the best of authors' knowledge, no one has reported an analytic solution valid for all dimensionless time $0 \le \tau < \infty$ in the whole spatial region $0 \le \eta < \infty$.

2. Mathematical description

Liao [23] solved the steady magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet by means of the homotopy analysis method. Here, we further consider the unsteady flows of an electrically conducting fluid, obeying the power-law model in the presence of a transverse magnetic field, past a flat sheet lying on the plane $y = 0$. Two equal but opposite forces are applied along the x-axis so that the wall is stretched keeping the original fixed. The unsteady MHD flows of this kind of non-Newtonian fluids are governed by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} - \left(\frac{\sigma B_0^2}{\rho}\right) u, \tag{1b}$$

where u and v are the velocity components in the x- and y-directions, t denotes the time, ρ, σ, B_0 and τ_{xy} are the density, electrical conductivity, magnetic field and shear stress, respectively. The shear tensor is defined by the Ostwald-de-Wäle model:

$$\tau_{ij} = 2K(2D_{kl}D_{kl})^{(n-1)/2}D_{ij}, \tag{2}$$

where

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3}$$

denotes the rate of stretching tensor, K is the consistency coefficient and n is the power-law index. The initial and boundary conditions are:

$$t < 0: \quad u = v = 0, \quad \text{at } y \ge 0, \ -\infty < x < +\infty, \tag{4a}$$

$$t \ge 0: \quad u = Cx, \ v = 0, \quad \text{at } y = 0, \tag{4b}$$

$$t \ge 0: \quad u \to 0, \quad \text{as } y \to +\infty, \tag{4c}$$

where C is a positive constant. Let ψ denote the stream function ψ, satisfying:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{5}$$

Following Liao [24] and Nazar et al. [41], we use the similarity transformations:

$$\psi = \left(\frac{K\xi}{\rho C^{1-2n}}\right)^{1/(n+1)} x^{2n/(n+1)} F(\eta, \xi), \tag{6}$$

and

$$\eta = y\left(\frac{\rho C^{2-n}}{K\xi}\right)^{1/(n+1)} x^{(1-n)/(1+n)},$$
$$\xi = 1 - \exp(-\tau), \quad \tau = Ct. \tag{7}$$

Then, the governing Eqs. (1a) and (1b) become:

$$(1-\xi)\left(\frac{\eta}{n+1}\frac{\partial^2 F}{\partial \eta^2} - \xi\frac{\partial^2 F}{\partial \eta \partial \xi}\right) + n\left(-\frac{\partial^2 F}{\partial \eta^2}\right)^{n-1}\frac{\partial^3 F}{\partial \eta^3} + \xi\left[\frac{2n}{n+1}F\frac{\partial^2 F}{\partial \eta^2} - \left(\frac{\partial F}{\partial \eta}\right)^2 - M\frac{\partial F}{\partial \eta}\right] = 0, \tag{8a}$$

subject to the boundary conditions:

$$F(0,\xi) = 0, \quad \left.\frac{\partial F(\eta,\xi)}{\partial \eta}\right|_{\eta=0} = 1, \quad \left.\frac{\partial F(\eta,\xi)}{\partial \eta}\right|_{\eta\to+\infty} = 0, \tag{8b}$$

where $M = \sigma B_0^2/(\rho C)$ is the magnetic parameter.

When $\xi = 0$, corresponding to $\tau = 0$, we have from (8a) that

$$n\left(-\frac{\partial^2 F}{\partial \eta^2}\right)^{n-1}\frac{\partial^3 F}{\partial \eta^3} + \frac{\eta}{n+1}\frac{\partial^2 F}{\partial \eta^2} = 0, \tag{9a}$$

subject to the boundary conditions:

$$F(0,0)=0,\quad \left.\frac{\partial F(\eta,\xi)}{\partial \eta}\right|_{\eta=0,\xi=0}=1,\quad \left.\frac{\partial F(\eta,\xi)}{\partial \eta}\right|_{\eta\to+\infty,\xi=0}=0. \tag{9b}$$

Especially, when $\xi=0$ and $n=1$, Eq. (9a) becomes the Rayleigh type of equation, and it has the exact solution:

$$F(\eta,0)=\eta\,\mathrm{erfc}(\eta/2)+\frac{2}{\sqrt{\pi}}[1-\exp(-\eta^2/4)]. \tag{10}$$

When $\xi=1$, corresponding to $\tau\to+\infty$, we have from (8a) that

$$n\left(-\frac{\partial^2 F}{\partial \eta^2}\right)^{n-1}\frac{\partial^3 F}{\partial \eta^3}+\frac{2n}{n+1}F\frac{\partial^2 F}{\partial \eta^2}-\left(\frac{\partial F}{\partial \eta}\right)^2-M\frac{\partial F}{\partial \eta}=0, \tag{11a}$$

subject to the boundary conditions:

$$F(0,1)=0,\quad \left.\frac{\partial F(\eta,\xi)}{\partial \eta}\right|_{\eta=0,\xi=1}=1,\quad \left.\frac{\partial F(\eta,\xi)}{\partial \eta}\right|_{\eta\to+\infty,\xi=1}=0. \tag{11b}$$

When $\xi=1$ and $n=1$, Eq. (11a) has the exact solution:

$$F(\eta,1)=\frac{1-\exp(-\sqrt{1+M}\eta)}{\sqrt{1+M}}. \tag{12}$$

For details, please refer to Liao [23] for the steady-state flows, corresponding to $\xi=1$.

The skin friction $C_f(\xi)$ at the wall is given by:

$$C_f(\xi)=\frac{\tau_w}{\rho(Cx)^2}=\xi^{-n/(n+1)}[-F_{\eta\eta}(0,\xi)]^n Re^{-1/(1+n)}, \tag{13}$$

where $Re=(Cx)^{2-n}x^n/(K/\rho)$ is the local Reynolds number.

3. Homotopy analysis method

3.1. Zeroth-order deformation equation

The steady-state magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet was solved by Liao [23]. Recently, by means of homotopy analysis method, Liao [24] obtained accurate solutions of a kind of unsteady boundary layer flows of a Newtonian fluid caused by an impulsively stretching flat plate. Following Liao [23,24], we express $F(\eta,\xi)$ by a set of base functions:

$$\{\xi^k\eta^m\exp(-n\eta)\,|\,k\geq 0,n\geq 0,m\geq 0\} \tag{14}$$

in the form:

$$F(\eta,\xi)=\sum_{k=0}^{+\infty}\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}a_{m,n}^k\xi^k\eta^m\exp(-n\eta), \tag{15}$$

where $a_{m,n}^k$ is a coefficient. This provides us with the so-called *Rule of Solution Expressions* for $F(\eta,\xi)$. According to the Rule of Solution Expression (15) and from (8a) and (8b), it is straightforward to choose:

$$F_0(\eta,\xi)=1-\exp(-\eta) \tag{16}$$

as the initial approximation of $F(\eta,\xi)$, and besides to choose:

$$\mathcal{L}[\Phi(\xi,\eta;q)]=\frac{\partial^3\Phi}{\partial\eta^3}-\frac{\partial\Phi}{\partial\eta} \tag{17}$$

as the auxiliary linear operator, which has the following property:

$$\mathcal{L}[C_1\exp(-\eta)+C_2\exp(\eta)+C_3]=0, \tag{18}$$

where C_1, C_2, and C_3 are constants. Based on (8a), we are led to define the nonlinear operator:

$$\mathcal{N}[\Phi(\eta,\xi;q)]=(1-\xi)\left(\frac{\eta}{n+1}\frac{\partial^2\Phi}{\partial\eta^2}-\xi\frac{\partial^2\Phi}{\partial\xi\partial\eta}\right)+n\left(-\frac{\partial^2\Phi}{\partial\eta^2}\right)^{n-1}\frac{\partial^3\Phi}{\partial\eta^3}+\xi\left[\frac{2n}{n+1}\Phi\frac{\partial^2\Phi}{\partial\eta^2}-\left(\frac{\partial\Phi}{\partial\eta}\right)^2-M\frac{\partial\Phi}{\partial\eta}\right]. \tag{19}$$

Let denote a non-zero auxiliary parameter. We construct the so-called zeroth-order deformation equation:

$$(1-q)\mathcal{L}[\Phi(\eta,\xi;q)-F_0(\eta,\xi)]=q\ \mathcal{N}[\Phi(\eta,\xi;q)], \tag{20a}$$

subject to the boundary conditions:

$$\Phi(0,\xi;q)=0,\quad \left.\frac{\partial\Phi(\eta,\xi;q)}{\partial\eta}\right|_{\eta=0}=0,\quad \left.\frac{\partial\Phi(\eta,\xi;q)}{\partial\eta}\right|_{\eta=+\infty}=0, \tag{20b}$$

where $q\in[0,1]$ is an embedding parameter.

Obviously, when $q=0$ and $q=1$, the above zeroth-order deformation Eqs. (20a) and (20b) have the solutions:

$$\Phi(\eta,\xi;0)=F_0(\eta,\xi), \tag{21}$$

and

$$\Phi(\eta,\xi;1)=F(\eta,\xi), \tag{22}$$

respectively. Thus as q increases from 0 to 1, $\Phi(\eta,\xi;q)$ varies from the initial guess $F_0(\eta,\xi)$ to the solution $F(\eta,\xi)$ of the considered unsteady problem. So, expanding $\Phi(\eta,\xi;q)$ in

Taylor's series with respect to the embedding parameter q, we have:

$$\Phi(\eta,\xi;q)=\Phi(\eta,\xi,0)+\sum_{m=1}^{+\infty}F_m(\eta,\xi)q^m, \tag{23}$$

where

$$F_m(\eta,\xi)=\frac{1}{m!}\left.\frac{\partial^m\Phi(\eta,\xi;q)}{\partial q^m}\right|_{q=0}. \tag{24}$$

Note that (20a) contains the auxiliary parameter . Assuming that is properly chosen so that the series (23) is convergent at $q=1$, we have, using (21) and (22), the solution series:

$$F(\eta,\xi)=F_0(\eta,\xi)+\sum_{m=1}^{+\infty}F_m(\eta,\xi). \tag{25}$$

3.2. High-order deformation equation

For the sake of simplicity, we define the vector:

$$\vec{F}_m=\{F_0,F_1,F_2,\ldots,F_m\}. \tag{26}$$

Differentiating the zeroth-order deformation equations (20a) m times with respect to q, then setting $q=0$, and finally dividing them by $m!$, we obtain the mth-order deformation equations:

$$\mathcal{L}[F_m(\eta,\xi)-\chi_m F_{m-1}(\eta,\xi)]=R_m(\vec{F}_{m-1}), \tag{27a}$$

subject to the boundary conditions:

$$F_m(0,\xi)=0,\quad \left.\frac{\partial F_m(\eta,\xi)}{\partial\eta}\right|_{\eta=0}=0,\quad \left.\frac{\partial F_m(\eta,\xi)}{\partial\eta}\right|_{\eta=+\infty}=0, \tag{27b}$$

where

$$R_m(\vec{F}_{m-1})=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathcal{N}[\Phi(\eta,\xi;q)]}{\partial q^{m-1}}\right|_{q=0} \tag{28}$$

and

$$\chi_m=\begin{cases}0, m=1,\\ 1, m>1.\end{cases} \tag{29}$$

Note that $R_m(\vec{F}_{m-1})$ depends on the integer power-law index n. When $n=1$, we have:

$$R_m(\vec{F}_{m-1})=(1-\xi)\left(\frac{\eta}{2}\frac{\partial^2F_{m-1}}{\partial\eta^2}-\xi\frac{\partial^2F_{m-1}}{\partial\eta\partial\xi}\right)+\frac{\partial^3F_{m-1}}{\partial\eta^3}+\xi\left[\sum_{i=0}^{m-1}F_i\frac{\partial^2F_{m-1-i}}{\partial\eta^2}-\sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-i-1}}{\partial\eta}-M\frac{\partial F_{m-1}}{\partial\eta}\right], \tag{30}$$

when $n=2$, it holds

$$R_m(\vec{F}_{m-1})=(1-\xi)\left(\frac{\eta}{3}\frac{\partial^2F_{m-1}}{\partial\eta^2}-\xi\frac{\partial^2F_{m-1}}{\partial\eta\partial\xi}\right)-2B_{m-1}+\xi\left[\sum_{i=0}^{m-1}\frac{4}{3}F_i\frac{\partial^2F_{m-1-i}}{\partial\eta^2}-\sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-i-1}}{\partial\eta}-M\frac{\partial F_{m-1}}{\partial\eta}\right], \tag{31}$$

when $n=3$, it reads

$$R_m(\vec{F}_{m-1})=(1-\xi)\left(\frac{\eta}{4}\frac{\partial^2F_{m-1}}{\partial\eta^2}-\xi\frac{\partial^2F_{m-1}}{\partial\eta\partial\xi}\right)+3\sum_{i=0}^{m-1}A_i\frac{\partial^3F_{m-1-i}}{\partial\eta^3}+\xi\left[\sum_{i=0}^{m-1}\frac{3}{2}F_i\frac{\partial^2F_{m-1-i}}{\partial\eta^2}-\sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-i-1}}{\partial\eta}-M\frac{\partial F_{m-1}}{\partial\eta}\right], \tag{32}$$

and when $n=4$, we have:

$$R_m(\vec{F}_{m-1})=(1-\xi)\left(\frac{\eta}{5}\frac{\partial^2F_{m-1}}{\partial\eta^2}-\xi\frac{\partial^2F_{m-1}}{\partial\eta\partial\xi}\right)-4\sum_{i=0}^{m-1}A_iB_{m-1-i}+\xi\left[\sum_{i=0}^{m-1}\frac{8}{5}F_i\frac{\partial^2F_{m-1-i}}{\partial\eta^2}-\sum_{i=0}^{m-1}\frac{\partial F_i}{\partial\eta}\frac{\partial F_{m-i-1}}{\partial\eta}-M\frac{\partial F_{m-1}}{\partial\eta}\right], \tag{33}$$

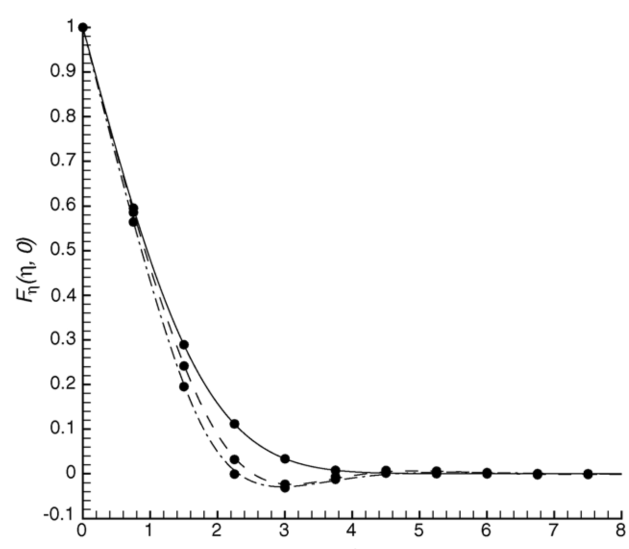

Fig. 1. The comparison of $F_\eta(\eta,0)$ of the analytic approximation with the exact and numerical solutions when $n=1,2,3$: (filled circles) 20th-order HAM approximations; (solid line) exact solution (10) when $n=1$; (dashed line) numerical solution when $n=2$; (dash-dotted line) numerical solution when $n=3$.

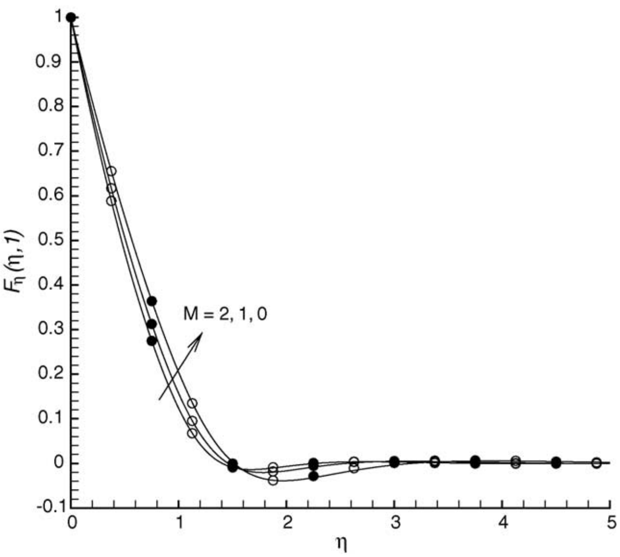

Fig. 2. The comparison of $F_\eta(\eta, 1)$ of the analytic approximations with the exact solution (12) and Liao's steady-state results [23] when $M = 0, 1, 2$ and $n = 1$: (filled circles) 20th-order HAM approximations; (open circles) steady-state solution given by Liao [23]; (lines) exact solution (12).

respectively, where

$$A_j = \sum_{i=0}^{j} \frac{\partial^2 F_i}{\partial \eta^2} \frac{\partial^2 F_{j-i}}{\partial \eta^2}, \tag{34}$$

$$B_j = \sum_{i=0}^{j} \frac{\partial^2 F_i}{\partial \eta^2} \frac{\partial^3 F_{j-i}}{\partial \eta^3}. \tag{35}$$

Let $F_m^*(\eta, \xi)$ denote a special solution of Eqs. (27a) and (27b). Using (18), we have its general solution:

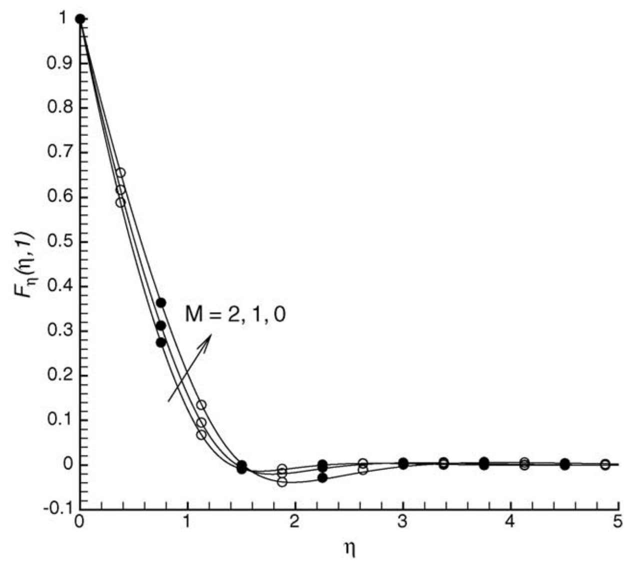

Fig. 4. The comparison of $F_\eta(\eta, 1)$ of the analytic approximations with the numerical solutions and Liao's steady-state results [23] when $M = 0, 1, 2$ and $n = 3$: (filled circles) 20th-order HAM approximations; (open circles) steady-state solution given by Liao [23]; (lines) numerical solution.

$$F_m(\eta, \xi) = F_m^*(\eta, \xi) + C_1 \exp(-\eta) + C_2 \exp(\eta) + C_3, \tag{36}$$

where the coefficients C_1, C_2, and C_3 are determined by the boundary conditions (27b), i.e.:

$$C_2 = 0, \quad C_1 = \left. \frac{\partial F_m^*(\eta, \xi)}{\partial \eta} \right|_{\eta=0}, \quad C_3 = -C_1 - F_m^*(0, \xi). \tag{37}$$

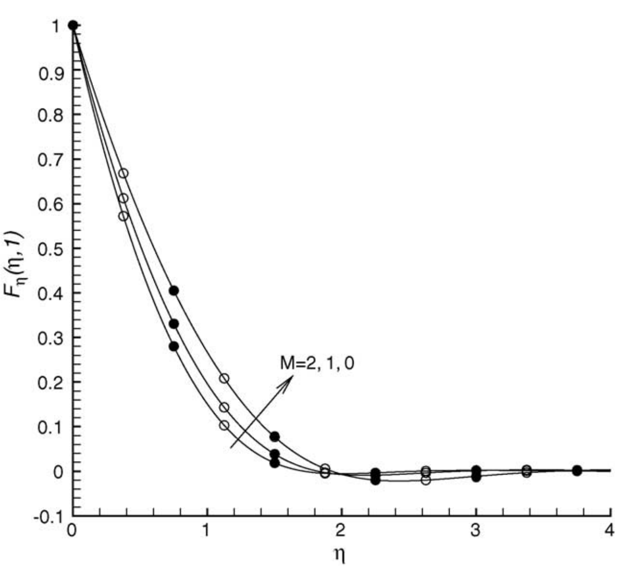

Fig. 3. The comparison of $F_\eta(\eta, 1)$ of the analytic approximations with the numerical solutions and Liao's steady-state results [23] when $M = 0, 1, 2$ and $n = 2$: (filled circles) 20th-order HAM approximations; (open circles) steady-state solution given by Liao [23]; (lines) numerical solution.

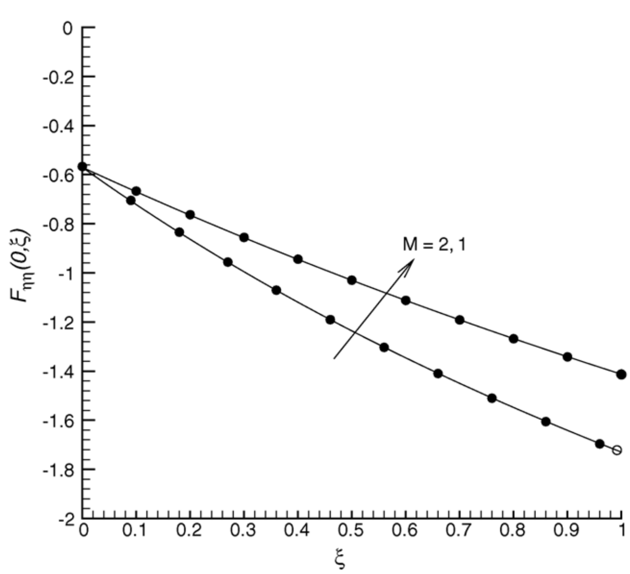

Fig. 5. The analytic approximations of $F_{\eta\eta}(0, \xi)$ for $0 \le \xi \le 1$ when $n = 1$ given by homotopy analysis method: (solid line) 16th-order approximation; (filled circles) 20th-order approximation; (open circles) steady-state solution given by Liao [23].

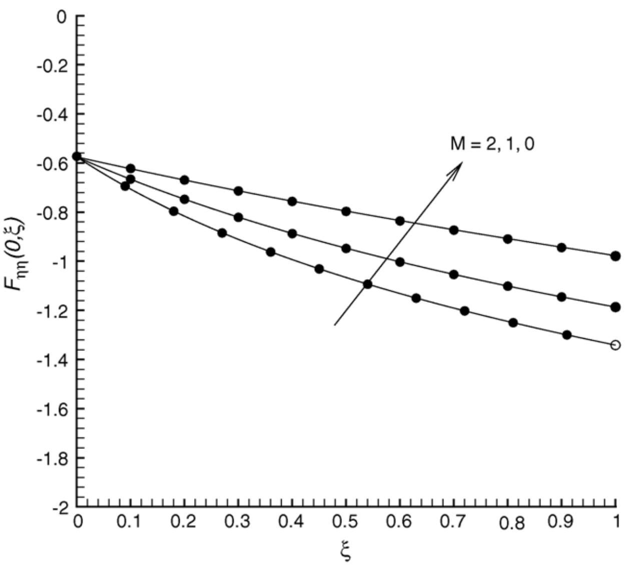

Fig. 6. The analytic approximations of $F_{\eta\eta}(0, \xi)$ for $0 \leq \xi \leq 1$ when $n = 2$ given by homotopy analysis method: (solid line) 16th-order HAM approximation; (filled circles) 20th-order HAM approximation; (open circles) steady-state solution given by Liao [23].

In this way, it is easy to solve the *linear* Eqs. (27a) and (27b) one after the other in the order $m = 1, 2, 3, \ldots$ by means of the symbolic computation software such as Mathematica.

4. Analysis of results

Liao [30] proved in general that, as long as a solution series given by the homotopy analysis method converges, it must be one of solutions. Note that the solution series (25) contains

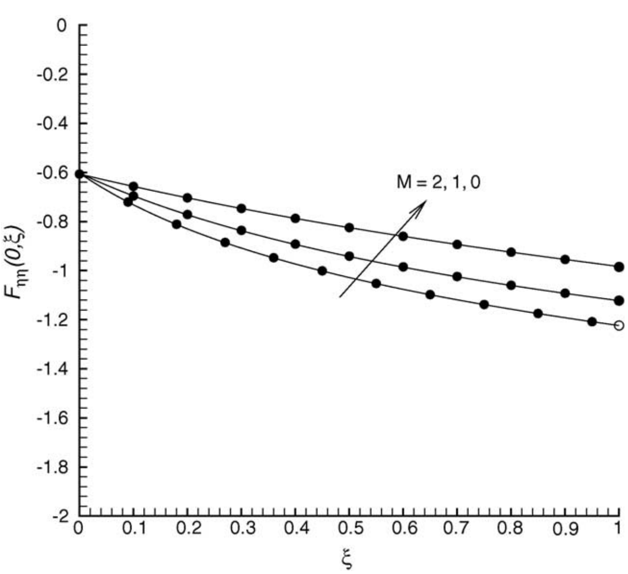

Fig. 7. The analytic approximations of $F_{\eta\eta}(0, \xi)$ for $0 \leq \xi \leq 1$ when $n = 3$ given by homotopy analysis method: (solid line) 16th-order HAM approximation; (filled circles) 20th-order HAM approximation; (open circles) steady-state solution given by Liao [23].

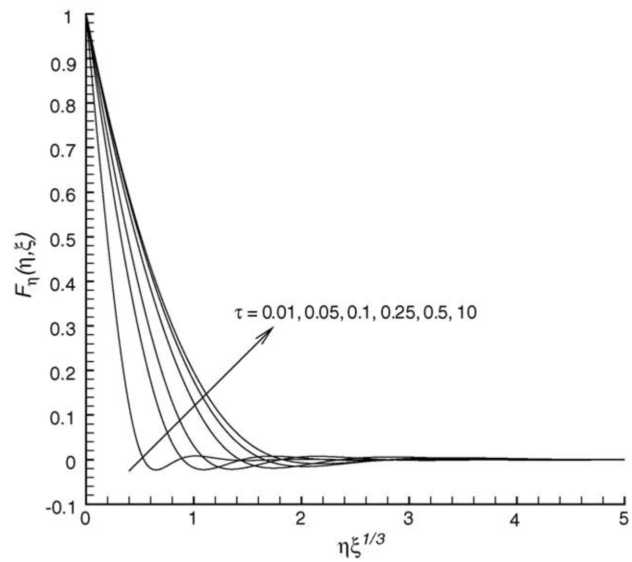

Fig. 8. The variation of the velocity profile $F_\eta(\eta, \xi)$ when $n = 2$ and $M = 1$.

the auxiliary parameter , for which we can choose a proper value by plotting so-called -curves to ensure that the solution series (25) converges, as suggested by Liao [30].

When $\xi = 0$, our analytic solutions agree well with the exact solution (10) when $n = 1$ and numerical results when $n = 2$ and 3, as shown in Fig. 1. When $n = 1$ and $\xi = 1$, corresponding to the steady-state Newtonian fluid, our analytic solution agrees well with the exact solution (12) and also Liao's steady-state results [23], as shown in Fig. 2. These verify the validation of the proposed analytic approach. When $n > 1$, corresponding to the non-Newtonian fluid, our analytic solutions agree well with the numerical solutions at $\xi = 0$ (the initial state) and $\xi = 1$ (the steady state), as shown in Figs. 1, 3 and 4. Note that all of these analytic results are obtained by means of $= -1/4$. Note also that, our analytic

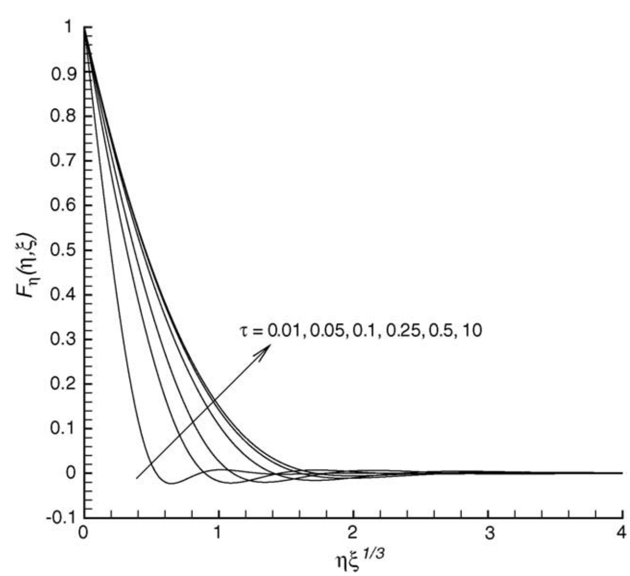

Fig. 9. The variation of the velocity profile $F_\eta(\eta, \xi)$ when $n = 2$ and $M = 2$.

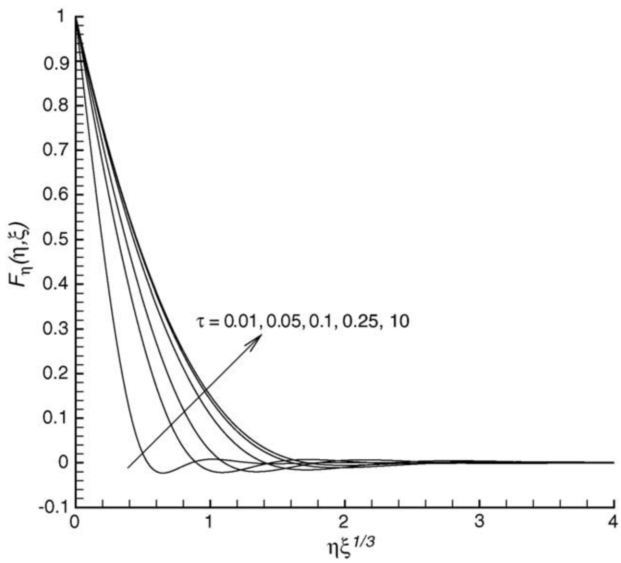

Fig. 10. The variation of the velocity profile $F_\eta(\eta, \xi)$ when $n = 3$ and $M = 1$.

results at $\xi = 1$ agree well with Liao's steady-state solutions [23], as shown in Figs. 3 and 4.

In a similar way, it is found that by means of $= -1/4$ the solution series (25) is convergent in the whole range of the dimensionless time $\xi \in [0, 1]$ for all considered cases of n and M, as shown in Figs. 5–7. Note that, as $\tau \to +\infty$, i.e. $\xi \to 1$, our analytic solutions tend to Liao's steady-state results [23]. Thus, by means of homotopy analysis method, we obtain analytic series solutions which are accurate and uniformly valid for all dimensionless time $\xi \in [0, 1]$ in the whole spatial region $0 \leq \eta < +\infty$. Such kind of analytic solutions has never been reported, to the best of our knowledge.

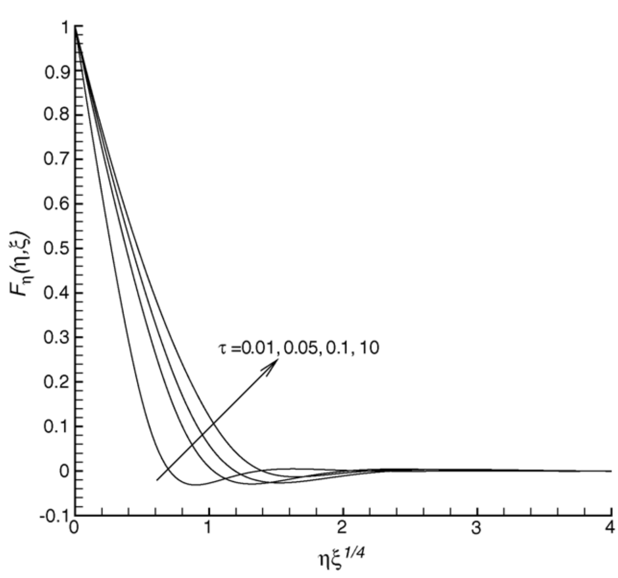

Fig. 11. The variation of the velocity profile $F_\eta(\eta, \xi)$ when $n = 3$ and $M = 2$.

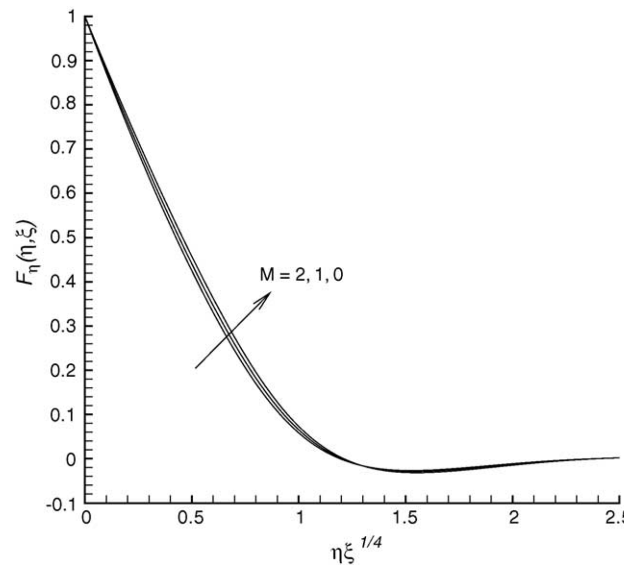

Fig. 12. The variation of the profile $F_\eta(\eta, \xi)$ when $n = 3$ and $M = 0, 1, 2$ at $\tau = 0.1$.

The variation of the velocity profiles as a function of τ for some different values of M and n is shown in Figs. 8–11. We can see that these velocity profiles develop rapidly from rest as τ increases from zero to ∞. The velocity profiles for the different values of M at the same dimensionless time $\tau = 0.1$, 0.5 when $n = 3$ are as shown in Figs. 12 and 13, respectively. Obviously, at any a given time, the velocity profile given by the larger value of M is closer to the corresponding initial ones. It seems that the flow for a larger magnetic parameter M develops more slowly. However, the transition from the unsteady initial flow up to the position where boundary layer start to separate is completely smooth for all values of M and n. For the same value of the magnetic parameter M, the

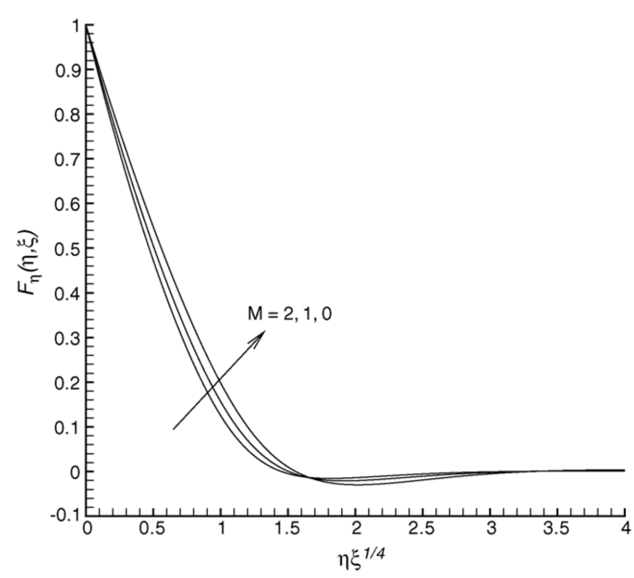

Fig. 13. The velocity profile $F_\eta(\eta, \xi)$ when $n = 3$ and $M = 0, 1, 2$ at $\tau = 0.5$.

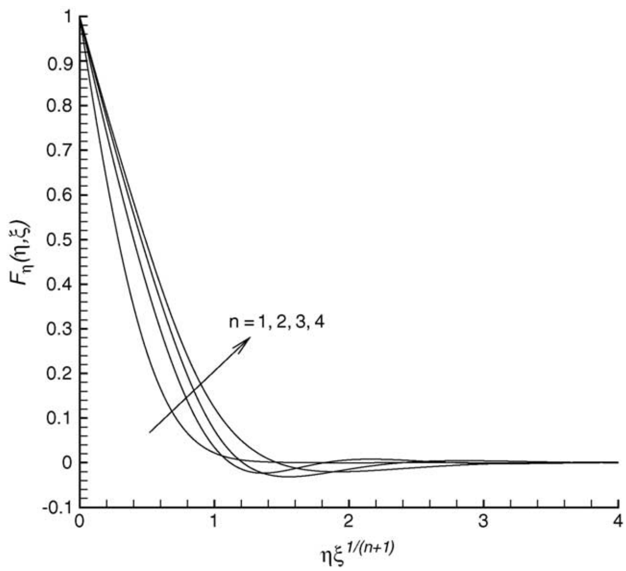

Fig. 14. The velocity profile $F_\eta(\eta, \xi)$ when $M = 0$ and $n = 1, 2, 3, 4$ at $\tau = 0.1$.

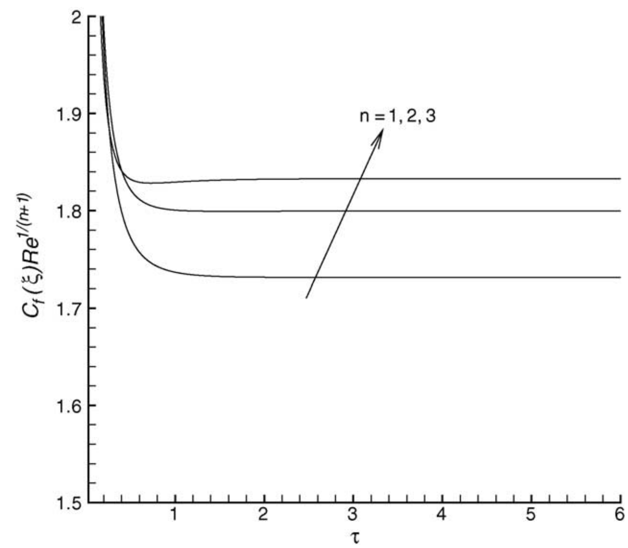

Fig. 16. The variation of the skin friction coefficient as a function of τ for the different power-law index n when $M = 2$.

velocity profiles for different n at the dimensionless time $\tau = 0.1$ are as shown in Figs. 14 and 15. Note that as n enlarges, the velocity profiles trend to the steady state more quickly.

The curves of the local skin friction coefficient $C_f(\xi)$ versus τ for a fixed value of either the power-law index n or the magnetic parameter M are as shown in Figs. 16 and 17, respectively. Note that, at the same dimensionless time $\tau \in (0, +\infty)$ and for the same power-law index n, the skin friction coefficient increases as the values of the magnetic parameter M enlarges. And for fixed values of M and the dimensionless time τ, the skin friction coefficient increases as the values of the power-law index n decreases. For steady-state flows, Liao [23] concluded that (A) the magnetic field tends to increase the wall friction, (B) this effect is more pronounced for sheared-thinning ($n < 1$) than for shear-thickening ($n > 1$) fluids. This paper indicates that Liao's conclusion (A) holds for all dimensionless time $0 \leq \tau < +\infty$. However, Liao's conclusion (B), which is based on the variation of the term $-F_{\eta\eta}(0, \xi)$, is not correct, because the wall friction is directly proportional to the term $[-F_{\eta\eta}(0, \xi)]^n$ but not to the term $-F_{\eta\eta}(0, \xi)$.

Thus, by means of homotopy method, we obtain the analytic series solutions which are accurate and uniformly valid for all dimensionless time $0 \leq \tau < \infty$ in the whole spatial region $0 \leq \eta < \infty$. To the best of our knowledge, such a kind of analytic solutions has never been reported.

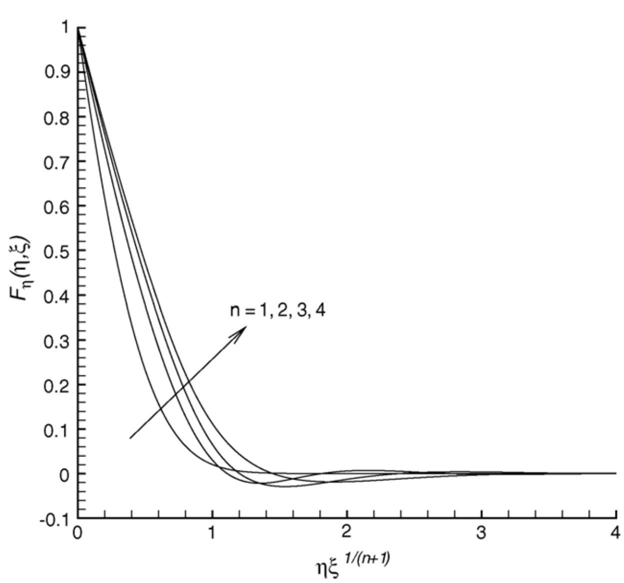

Fig. 15. The velocity profile $F_\eta(\eta, \xi)$ when $M = 1$ and $n = 1, 2, 3, 4$ at $\tau = 0.1$.

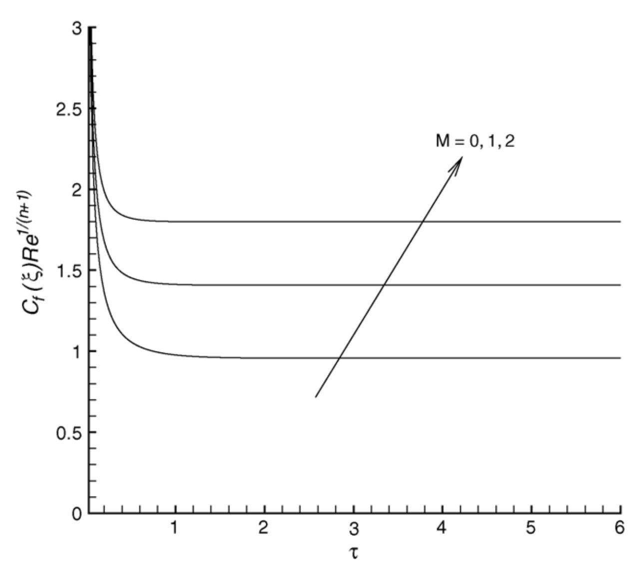

Fig. 17. The variation of the skin friction coefficient as a function of τ for the different parameter M when $n = 2$.

5. Conclusions

In this paper, we apply the homotopy analysis method to study the unsteady magnetohydrodynamic (MHD) viscous flows of non-Newtonian fluids caused by an impulsively stretching plate. Different from previous analytic solutions, our series solutions are valid for *all* dimensionless time $0 \le \tau < \infty$ in the whole spatial region $0 \le \eta < \infty$. To the best of our knowledge, such kind of analytic solutions has never been reported. The effects of the integral power-law index ($n = 1, 2, 3, 4$) of the non-Newtonian fluids and the magnetic parameter $M = 0, 1, 2$ on the flows are investigated. We show that, in the whole dimensionless time $0 \le \tau < +\infty$, the magnetic field tends to increase the wall friction, and that this effect is more pronounced for non-Newtonian fluids with larger power-law index.

The proposed analytic approach has general meaning and thus may be applied in a similar way to other unsteady non-linear problems to get accurate analytic solutions valid for all dimensionless time.

Acknowledgements

Thanks to the anonymous reviewers for their valuable suggestions. This work is supported by "National Science Fund for Distinguished Young Scholars" (Approval No. 50125923) of Natural Science Foundation of China for the financial support. The corresponding author would like to express his sincere acknowledgement to Li Ka Shing Foundation for the financial support (Cheung Kong Programme).

References

[1] B.C. Sakiadis, Boundary-layer behavior on continuous solid surface. I. Boundary-layer equations for two-dimensional and axisymmetric flow, AIChE J. 7 (1961) 26–28.
[2] F.K. Tsou, E.M. Sparrow, R.J. Glodstein, Flow and heat transfer in the boundary layer on a continuous moving surface, Int. J. Heat Mass Transf. 10 (1967) 219–235.
[3] L.E. Erickson, L.T. Fan, V.G. Fox, Heat and mass transfer on a moving continuous flat plate with suction or injection, Ind. Eng. Chem. 5 (1966) 19–25.
[4] T.S. Chen, F.A. Strobel, Buoyancy effects in boundary layer adjacent to a continuous, moving horizontal flat plate, J. Heat Trans-T ASME 102 (1980) 170–172.
[5] A.M. Jacobi, A scale analysis approach to rhe correlation of continuous moving sheet (backward boundary layer) forced convective heat transfer, J. Heat Trans-T ASME 115 (1993) 1058–1061.
[6] L.J. Crane, Flow past a stretching plane, Z. Angew. Math. Phys. 21 (1970) 645–647.
[7] C.K. Chen, M. Char, Heat transfer of a continuous stretching surface with suction or blowing, J. Math. Anal. Appl. 135 (1988) 568–580.
[8] P.S. Gupta, A.S. Gupta, Heat and mass transfer on a stretching sheet with suction or blowing, Can. J. Chem. Eng. 55 (1977) 744–746.
[9] K.R. Rajagopal, A non-similar boundary layer on a stretching sheet in a non-Newtonian fluid with uniform free stream, J. Math. Phys. Sci. 21 (1987) 189–200.
[10] W.C. Troy, E.A. Overman, G.B. Ermentrout, J.P. Keener, Uniqueness of flow of a second-order fluid past a stretching surface, Q. Appl. Math. 44 (1987) 753–755.
[11] A. Chakrabarti, A.S. Gupta, Hydromagnetic flow and heat transfer over a stretching sheet, Q. Appl. Math. 37 (1979) 73–78.
[12] K. Vajravelu, A. Hadjinicolaou, Convective heat transfer in an electrically conducting fluid at a stretching surface with uniform free stream, Int. J. Eng. Sci. 35 (1997) 1237–1244.
[13] A.J. Chamkha, Hydromagnetic three-dimensional free convection on a vertical stretching surface with heat generation or absorption, Int. J. Heat Fluid Flow 20 (1999) 84–92.
[14] H.S. Takhar, A.A. Raptis, A.A. Perdikis, MHD asymmetric flow past a semi-infinite moving plate, Acta Meth. 65 (1987) 287–290.
[15] K. Vajravelu, Hydromagnetic flow and heat transfer over a continuous, moving, porous, flat surface, Acta Meth. 64 (1986) 179–185.
[16] H.I. Andersson, MHD flow of a viscous fluid past a stretching surface, Acta Meth. 95 (1992) 227–230.
[17] H.I. Andersson, A exact solution of the Navier-Stokes equations for magnetohydrodynamics flow, Acta Meth. 113 (1995) 241–244.
[18] H. Xu, An explicit analytic solution for convective heat transfer in an electrically conducting fluid at a stretching surface with uniform free stream, Int. J. Eng. Sci., in press.
[19] T. Sarpkaya, Flow of non-Newtonian fluids in a magnetic filed, AIChE J. 7 (1961) 324–328.
[20] D.S. Djukic, On the use of Crocco's equation for flow of power-law fluids in a transverse magnetic field, AIChE J. 19 (1973) 1159–1163.
[21] D.S. Djukic, Hiemenz magnetic flow of power-law fluids, J. Appl. Mech-T ASME 41 (1974) 822–823.
[22] H.I. Andersson, K.H. Bech, B.S. Dandapat, Magnetohydrodynamic flow of a power-law fluid over a stretching sheet, Int. J. Non-linear Mech. 27 (1992) 929–936.
[23] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189–212.
[24] S.J. Liao, An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate, Commun. Nonlinear Sci. Num. Simulat., in press.
[25] W.T. Cheng, C.N. Huang, Unsteady flow and heat transfer on an accelerating surface with blowing or suction in the absence and presence of a heat source or sink, Chem. Eng. Sci. 59 (2004) 771–780.
[26] P.J. Hilton, An Introduction to Homotopy Theory, Cambridge University Press, 1953.
[27] S. Sen, Topology and Geometry for Physicists, Academic Press, Florida, 1983.
[28] E.I. Grigolyuk, V.I. Shalashilin, Problems of Nonlinear Deformation: The Continuation Method Applied to Nonlinear Problems in Solid Mechanics, Kluwer Academic Publishers, Dordrecht, 1991.
[29] J.C. Alexander, J.A. Yorke, The homotopy continuation method: numerically implementable topological procedures, Trans. Am. Math. Soc. 242 (1978) 271–284.
[30] S.J. Liao, Beyond Perturbation: Introduction to Homotopy Analysis Method, Chapman & Hall/CRC Press, Boca Raton, 2003.
[31] A.H. Nayfeh, Perturbation Methods, John Wiley & Sons, New York, 2000.
[32] A.M. Lyapunov, General Problem on Stability of Motion, Taylor & Francis, London, 1992 (English translation).
[33] A.V. Karmishin, A.T. Zhukov, V.G. Kolosov, Methods of Dynamics Calculation and Testing for Thin-walled Structures, Mashinostroyenie, Moscow, 1990 (in Russian).
[34] G. Adomian, Nonlinear Stochastic Differential Equations, J. Math. Anal. Appl. 55 (1975) 441–452.
[35] S.J. Liao, A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate, J. Fluid Mech. 385 (1999) 101–128.
[36] S.J. Liao, An explicit, totally analytic approximate solution for Blasius' viscous flow problems, Int. J. Non-linear Mech. 34 (1999) 759–778.

[37] S.J. Liao, An analytic approximation of the drag coefficient for the viscous flow past a sphere, Int. J. Non-linear Mech. 37 (2002) 1–18.
[38] S.J. Liao, A. Campo, Analytic solutions of the temperature distribution in Blasius viscous flow problems, J. Fluid Mech. 453 (2002) 411–425.
[39] S.J. Liao, K.F. Cheung, Homotopy analysis of nonlinear progressive waves in deep water, J. Eng. Math. 45 (2003) 105–116.
[40] H. Xu, An expilcit analytic solution for free convection about a vertical flat plate embedded in a porous medium by means of homotopy analysis method, Appl. Math. Comp. 158 (2004) 433–443.
[41] N. Nazar, N. Amin, I. Pop, Unsteady boudary layer flow due to stretching surface in a rotating fluid, Mech. Res. Commun. 31 (2004) 121–128.