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Analytic approximations of Von Kármán plate under arbitrary uniform pressure—equations in integral form

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Analytic approximations of the Von Kármán's plate equations in integral form for a circular plate under external uniform pressure to arbitrary magnitude are successfully obtained by means of the homotopy analysis method (HAM), an analytic approximation technique for highly nonlinear problems. Two HAM-based approaches are proposed for either a given external uniform pressure Q or a given central deflection, respectively. Both of them are valid for uniform pressure to arbitrary magnitude by choosing proper values of the so-called convergence-control parameters c_1 and c_2 in the frame of the HAM. Besides, it is found that the HAM-based iteration approaches generally converge much faster than the interpolation iterative method. Furthermore, we prove that the interpolation iterative method is a special case of the first-order HAM iteration approach for a given external uniform pressure Q when $c_1 = -\theta$ and $c_2 = -1$, where θ denotes the interpolation iterative parameter. Therefore, according to the convergence theorem of Zheng and Zhou about the interpolation iterative method, the HAM-based approaches are valid for uniform pressure to arbitrary magnitude at least in the special case $c_1 = -\theta$ and $c_2 = -1$. In addition, we prove that the HAM approach for the Von Kármán's plate equations in differential form is just a special case of the HAM for the Von Kármán's plate equations in integral form mentioned in this paper. All of these illustrate the validity and great potential of the HAM for highly nonlinear problems, and its superiority over perturbation techniques.

circular plate, high nonlinearity, homotopy analysis method

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1 Introduction

The Von Kármán's plate equations [1, 2] in integral form describing the large deflection of a circular thin plate under uniform pressure read

$$\mathcal{N}_1[\varphi(y), S(y)] = \varphi(y) + \int_0^1 \frac{1}{\varepsilon^2} K(y,\varepsilon) S(\varepsilon)\varphi(\varepsilon)\mathrm{d}\varepsilon + \int_0^1 K(y,\varepsilon) Q \mathrm{d}\varepsilon = 0, \tag{1}$$

$$\mathcal{N}_2[\varphi(y), S(y)] = S(y) - \frac{1}{2}\int_0^1 \frac{1}{\varepsilon^2} G(y,\varepsilon)\varphi^2(\varepsilon)\mathrm{d}\varepsilon = 0, \tag{2}$$

in which

$$K(y,\varepsilon) = \begin{cases} (\lambda-1)y\varepsilon + y, & y \le \varepsilon, \\ (\lambda-1)y\varepsilon + \varepsilon, & y > \varepsilon, \end{cases} \tag{3}$$

$$G(y,\varepsilon) = \begin{cases} (\mu-1)y\varepsilon + y, & y \le \varepsilon, \\ (\mu-1)y\varepsilon + \varepsilon, & y > \varepsilon, \end{cases} \tag{4}$$

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with the definitions

$$y = \frac{r^2}{R_a^2}, \quad W(y) = \sqrt{3(1-\nu^2)}\frac{w(y)}{h}, \quad \varphi(y) = y\frac{\mathrm{d}W(y)}{\mathrm{d}y}, \tag{5}$$

$$S(y) = 3(1-\nu^2)\frac{R_a^2 N_r}{Eh^3}y, \quad Q = \frac{3(1-\nu^2)\sqrt{3(1-\nu^2)}R_a^4}{4Eh^4}p, \tag{6}$$

where r is the radial coordinate whose origin locates at the center of the plate; the constants E, ν, R_a, h are elastic modulus, the Poisson's ratio, radius and thickness of the plate, respectively; $w(y)$ and N_r denote the deflection and the radial membrane force of the plate; p represents the external uniform pressure; λ and μ are parameters related to the boundary conditions at $y = 1$. From eq. (5), we have the dimensionless central deflection,

$$W(y) = -\int_y^1 \frac{1}{\varepsilon}\varphi(\varepsilon)\mathrm{d}\varepsilon. \tag{7}$$

Without loss of generality, let us consider here the large deflection of a circular thin plate with clamped boundary, say,

$$\lambda = 0, \quad \mu = 2/(1-\nu). \tag{8}$$

Besides, the Poisson's ratio ν is taken to be 0.3.

In 1958, Keller and Reiss [3] proposed the interpolation iterative method to solve the Von Kármán's plate equations in integral form by introducing an interpolation iterative parameter to the iteration procedure, and they successfully obtained convergent solutions for uniform pressure as high as $Q = 7000$. The iterative procedures of the interpolation iterative method [1, 3] for the Von Kármán's plate equations in integral form are as follows:

$$\begin{cases} \psi_n(y) = \dfrac{1}{2}\displaystyle\int_0^1 \frac{1}{\varepsilon^2}G(y,\varepsilon)\vartheta_n^2(\varepsilon)\mathrm{d}\varepsilon, \\ \vartheta_{n+1}(y) = (1-\theta)\vartheta_n(y) - \theta\displaystyle\int_0^1 K(y,\varepsilon)Q\mathrm{d}\varepsilon \\ \qquad -\theta\displaystyle\int_0^1 \frac{1}{\varepsilon^2}K(y,\varepsilon)\vartheta_n(\varepsilon)\psi_n(\varepsilon)\mathrm{d}\varepsilon, \end{cases} \tag{9}$$

with the definition of such an initial guess

$$\vartheta_1(y) = -\frac{Q\theta}{2}\left[(\lambda+1)y - y^2\right], \tag{10}$$

where θ is an interpolation iterative parameter. It should be emphasized that Zheng and Zhou [1,4] gave an elegant proof about the convergence of the interpolation iterative method, say, convergent solutions can be obtained by the interpolation iterative method for arbitrary value of uniform pressure Q if a proper interpolation iterative parameter is chosen.

The Von Kármán's plate equations [2] in differential form for a circular plate under external uniform pressure to arbitrary amplitude have been successfully solved [5] by means of the homotopy analysis method (HAM) [6-17], an analytic approximation method for highly nonlinear problems. By choosing a proper value of the so-called convergence-control parameter c_0, convergent series solutions for four types of boundary conditions were obtained even in the cases with rather high nonlinearity [5]. It is found that the convergence-control parameter c_0 plays an important role: it is the convergence-control parameter c_0 that guarantees the convergence of solution series, and thus distinguishes the HAM from other analytic methods. Besides, it is found that the iteration technique can greatly boost the computational efficiency. In addition, it was even proved in ref. [5] that perturbation methods for an arbitrary perturbation quantity (including Vincent's [18] and Chien's [19] perturbation methods) and the modified iteration method [20] are only the special cases of the HAM when $c_0 = -1$.

In history, the Von Kármán's plate is a very classical problem in nonlinear mechanics. So, in this paper, we further propose two approaches in the frame of the HAM to solve the Von Kármán's plate equations in integral form. It is found that convergent solution series can be obtained within a rather large ratio of central deflection to plate thickness $w(0)/h > 20$, even when $Q = 39000$, corresponding to a rather high nonlinearity. Especially, we prove that the interpolation iterative method [3] is also a special case of the HAM, too. Moreover, we further prove that the HAM for the two forms, i.e., differential and integral form, of the Von Kármán's plate equations [2] are equivalent if we choose the same initial guesses, but the HAM for the Von Kármán's plate equations in integral form is more general, since the initial guesses no longer have to satisfy the boundary conditions of the Von Kármán's plate equations in differential form. Therefore, according to the convergence theorem of Zheng and Zhou [1,4] about the interpolation iterative method, it is easy to understand why the HAM can guarantee the convergence of solution series for the two forms of the Von Kármán's plate equations under uniform pressure to arbitrary amplitude.

2 HAM approach for a given external uniform pressure Q to arbitrary amplitude

2.1 Mathematical formulas

Like ref. [5], we express $\varphi(y)$ and $S(y)$ as:

$$\varphi(y) = \sum_{k=1}^{+\infty} a_k \cdot y^k, \quad S(y) = \sum_{k=1}^{+\infty} b_k \cdot y^k, \tag{11}$$

where a_k and b_k are constant coefficients to be determined.

Let $\varphi_0(y)$ and $S_0(y)$ be initial guesses of $\varphi(y)$ and $S(y)$, respectively. Moreover, let c_1 and c_2 denote the non-zero auxiliary parameters, called the convergence-control parameters, and $q \in [0,1]$ the embedding parameter, respectively. Then we construct a family of equations in $q \in [0,1]$, namely the zeroth-order deformation equations

$$
\begin{aligned}
&(1-q)\Big[\Phi(y,q)-\varphi_0(y)\Big]\\
=&c_1 q\Big[\Phi(y,q)+\int_0^1 \frac{1}{\varepsilon^2}K(y,\varepsilon)\Phi(\varepsilon,q)\Xi(\varepsilon,q)\mathrm{d}\varepsilon\\
&+\int_0^1 K(y,\varepsilon)Q\mathrm{d}\varepsilon\Big],
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&(1-q)\Big[\Xi(y,q)-S_0(y)\Big]\\
=&c_2 q\left[\Xi(y,q)-\frac{1}{2}\int_0^1 \frac{1}{\varepsilon^2}G(y,\varepsilon)\Phi^2(\varepsilon,q)\mathrm{d}\varepsilon\right].
\end{aligned} \tag{13}
$$

When $q=0$, eqs. (12) and (13) have the solution

$$\Phi(y,0)=\varphi_0(y), \quad \Xi(y,0)=S_0(y). \tag{14}$$

When $q=1$, eqs. (12) and (13) are equivalent to the original eqs. (1) and (2), provided

$$\Phi(y,1)=\varphi(y), \quad \Xi(y,1)=S(y). \tag{15}$$

Therefore, as q increases from 0 to 1, $\Phi(y,q)$ varies continuously from the initial guess $\varphi_0(y)$ to $\varphi(y)$, so does $\Xi(y,q)$ from the initial guess $S_0(y)$ to $S(y)$.

Using eq. (14), we have the Maclaurin series with respect to the embedding parameter q:

$$
\begin{cases}
\Phi(y,q)=\varphi_0(y)+\displaystyle\sum_{k=1}^{+\infty}\varphi_k(y)\, q^k,\\
\Xi(y,q)=S_0(y)+\displaystyle\sum_{k=1}^{+\infty}S_k(y)\, q^k,
\end{cases} \tag{16}
$$

where

$$\varphi_k(y)=\mathcal{D}_k[\Phi(y,q)], \qquad S_k(y)=\mathcal{D}_k[\Xi(y,q)], \tag{17}$$

in which

$$\mathcal{D}_k[f]=\frac{1}{k!}\frac{\partial^k f}{\partial q^k}\bigg|_{q=0} \tag{18}$$

is called the kth-order homotopy-derivative of f.

Note that there are two convergence-control parameters c_1 and c_2 in the Maclaurin series (16). Assume that c_1 and c_2 are properly chosen so that the power series (16) converge at $q=1$. Then according to eq. (15), we have the so-called homotopy-series solutions

$$\varphi(y)=\sum_{k=0}^{+\infty}\varphi_k(y), \qquad S(y)=\sum_{k=0}^{+\infty}S_k(y). \tag{19}$$

Substituting the Maclaurin series (16) into the zeroth-order deformation eqs. (12) and (13), and then equating the like-power of the embedding parameter $q \in [0,1]$, we have the so-called kth-order deformation equations

$$\varphi_k(y)=\chi_k\varphi_{k-1}(y)+c_1\,\delta_{1,k-1}(y), \tag{20}$$

$$S_k(y)=\chi_k S_{k-1}(y)+c_2\,\delta_{2,k-1}(y), \tag{21}$$

where

$$
\begin{aligned}
&\delta_{1,k-1}(y)\\
=&\varphi_{k-1}(y)+(1-\chi_k)\int_0^1 K(y,\varepsilon)Q\mathrm{d}\varepsilon\\
&+\int_0^1 \frac{1}{\varepsilon^2}K(y,\varepsilon)\sum_{i=0}^{k-1}\varphi_i(\varepsilon)S_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon,
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&\delta_{2,k-1}(y)\\
=&S_{k-1}(y)-\frac{1}{2}\int_0^1 \frac{1}{\varepsilon^2}G(y,\varepsilon)\sum_{i=0}^{k-1}\varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon,
\end{aligned} \tag{23}
$$

with the definition

$$
\chi_k=\begin{cases}0, & \text{when } k\le 1,\\ 1, & \text{when } k>1.\end{cases} \tag{24}
$$

Note that, in the frame of the HAM, we have great freedom to choose the initial guesses $\varphi_0(y)$ and $S_0(y)$. But $\varphi_0(y)$ and $S_0(y)$ should satisfy the expression (11), thus, we choose the initial guesses:

$$\varphi_0(y)=\frac{Qc_0}{2}[(\lambda+1)y-y^2], \qquad S_0(y)=0. \tag{25}$$

Then, by means of eqs. (20) and (21), $\varphi_k(y)$ and $S_k(y)$ can be obtained step by step, starting from $k=1$. The nth-order approximations of $\varphi(y)$ and $S(y)$ read

$$\tilde{\varphi}(y)=\sum_{k=0}^{n}\varphi_k(y), \qquad \tilde{S}(y)=\sum_{k=0}^{n}S_k(y). \tag{26}$$

For the sake of simplicity, we set

$$c_1=c_2=c_0. \tag{27}$$

Define the squared residual error

$$\mathcal{E}=\int_0^1\left\{\left(\mathcal{N}_1\left[\tilde{\varphi}(y),\tilde{S}(y)\right]\right)^2+\left(\mathcal{N}_2\left[\tilde{\varphi}(y),\tilde{S}(y)\right]\right)^2\right\}\mathrm{d}y, \tag{28}$$

where the nonlinear operators defined by $\mathcal{N}_1$ and $\mathcal{N}_2$ are related to the original eqs. (1) and (2). Note that the squared residual error $\mathcal{E}$ is dependent upon the unknown convergence-control parameter c_0. Obviously, the smaller the $\mathcal{E}$, the more accurate the HAM approximations.

2.2 Results given by the non-iteration approach

First of all, the normal HAM approach without iteration is used to solve the Von Kármán's plate equations with the clamped boundary for a given external uniform pressure Q. Without loss of generality, let us consider the case of $Q = 5$. At the beginning, the so-called convergence-control parameter c_0 is unknown. Its optimal value (i.e., $c_0 = -0.35$ in this case) is determined by the minimum of the squared residual error $\mathcal{E}$ defined by eq. (28), although convergent results can be obtained for any $c_0 \in (-0.7, 0)$. According to Table 1, the squared residual error quickly decreases to 2×10^{-7} by means of $c_0 = -0.35$ in the case of $Q = 5$, corresponding to $w(0)/h = 0.62$. Note that Vincent's perturbation results [18] (using Q as the perturbation quantity) for a circular plate with the clamped boundary are only valid for $w(0)/h < 0.52$, corresponding to $Q < 3.9$, and thus fail in the case of $Q = 5$. So, the convergence control parameter c_0 indeed provides us a simple way to guarantee the convergence of solution series.

It is found in a similar way that, for a given value of Q, the optimal value of c_0 can be obtained, which can be expressed by such an empirical formula

$$c_0 = -\frac{13}{13 + Q^2} \quad (0 < Q \leq 5). \tag{29}$$

The convergent homotopy-approximations of $w(0)/h$ in case of different values of Q are given in Table 2. For larger value of Q, one can also gain the convergent series solution, but with more CPU times. In this case, the iteration is used to accelerate the convergence, as described below.

Table 1 The squared residual error $\mathcal{E}$ and the approximations of $w(0)/h$ in the case of $Q = 5$ by means of the HAM without iteration (see sect. 2.1) using $c_0 = -0.35$

m, order of approx.	$\mathcal{E}$	$w(0)/h$
10	3×10^{-4}	0.64
20	7×10^{-5}	0.62
30	1×10^{-5}	0.62
40	2×10^{-6}	0.62
50	2×10^{-7}	0.62

Table 2 The homotopy-approximations of $w(0)/h$ versus Q, given by the HAM-based approach without iteration (see sect. 2.1) using the optimal convergence-control parameter c_0 given by the empirical formula (29)

Q	c_0	$w(0)/h$
1	−0.93	0.15
2	−0.76	0.29
3	−0.59	0.41
4	−0.45	0.53
5	−0.34	0.62

2.3 Convergence acceleration by iteration

As shown in ref. [5], the convergence of the homotopy-series solutions can be greatly accelerated by means of iteration technique, so the Mth-order homotopy-approximations

$$\varphi^*(y) \approx \varphi_0(y) + \sum_{k=1}^{M} \varphi_k(y), \quad S^*(y) \approx S_0(y) + \sum_{k=1}^{M} S_k(y), \tag{30}$$

are used as the new initial guesses for the next iteration, say, $\varphi_0(y) = \varphi^*(y)$ and $S_0(y) = S^*(y)$. This provides us the Mth-order iteration approach of the HAM.

Without loss of generality, let us consider the case of $Q = 1000$. As shown in Figure 1, the higher the order M of iteration, the less times of iteration are required for a given accuracy-level of approximation, but the slower the approximation converges[1)]. Thus, from the view-point of computational efficiency, we choose the first-order iteration approach (i.e., $M = 1$). It is found that, the corresponding optimal convergence-control parameter c_0 can be expressed by the empirical formula:

$$c_0 = -\frac{23}{Q + 23}, \tag{31}$$

within the range of $Q \leq 1000$. The approximations of $w(0)/h$ in case of different values of Q are listed in Table 3.

2.4 Relations to the interpolation iterative method

Here we prove that the interpolation iterative method [3] is a special case of the first-order HAM iteration approach mentioned above in sect. 2.3.

Recall that θ denotes the interpolation iterative parameter in the interpolation iterative method [3]. By means of the first-order HAM-based iteration approach, the new approximations $\varphi^*(y) = \varphi_0(y) + \varphi_1(y)$ and $S^*(y) = S_0(y) + S_1(y)$ are used as the new initial guesses $\varphi_0(y)$ and $S_0(y)$ for the next iteration, since the HAM provides us the freedom to choose initial guesses.

Note that the HAM provides us freedom to choose the convergence-control parameters c_1 and c_2. So, let us choose $c_1 = -\theta$ and $c_2 = -1$. Then, according to eqs. (20) and (24), we have the first-order homotopy-approximations of $\varphi(y)$ and $S(y)$

$$\begin{aligned} S^*(y) &= S_0(y) + S_1(y) \\ &= S_0(y) - \delta_{2,0}(y) \\ &= \frac{1}{2} \int_0^1 \frac{1}{\varepsilon^2} G(y, \varepsilon) \varphi_0^2(\varepsilon) d\varepsilon, \end{aligned} \tag{32}$$

1) All examples considered in this paper are computed using a laptop, which has a core i7 3.60 GHz process with 8 GB memory.

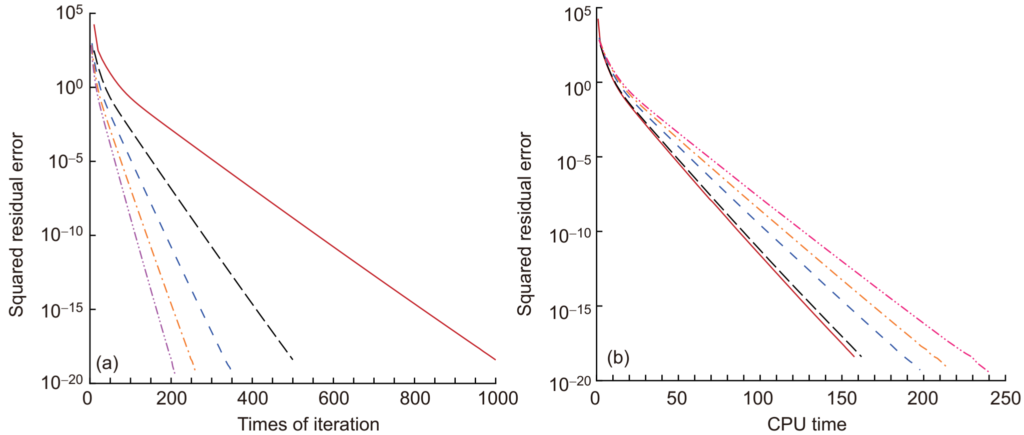

Figure 1 (Color online) The squared residual error $\mathcal{E}$ versus the times of iteration (a) and the CPU times (b) in the case of $Q = 1000$, given by the HAM-based iteration approach using the convergence-control parameter $c_0 = -0.02$. Solid line: first-order; long-dashed line: second-order; dashed line: third-order; dash-dotted line: fourth-order; dash-double-dotted line: fifth-order.

Table 3 The homotopy-approximations of $w(0)/h$ in case of different values of Q, given by the first-order HAM iteration approach using the optimal convergence-control parameter c_0 given by eq. (31)

Q	c_0	$w(0)/h$
200	−0.10	3.5
400	−0.05	4.5
600	−0.04	5.2
800	−0.03	5.7
1000	−0.02	6.1

$$\begin{aligned}\varphi^*(y) =& \varphi_0(y) + \varphi_1(y)\\ =& \varphi_0(y) - \theta\delta_{1,0}(y)\\ =& (1-\theta)\varphi_0(y) - \theta\int_0^1 K(y,\varepsilon)Q\mathrm{d}\varepsilon\\ &- \theta\int_0^1 \frac{1}{\varepsilon^2}K(y,\varepsilon)\varphi_0(\varepsilon)S_0(\varepsilon)\mathrm{d}\varepsilon.\end{aligned} \quad (33)$$

Since the initial guesses are given at the beginning, we take the following iterative procedures:

(A) Calculate $S^*(y)$ according to eq. (32);

(B) replace $S_0(y)$ by $S^*(y)$ as the new initial guess, i.e., $S_0(y) = S^*(y)$;

(C) calculate $\varphi^*(y)$ according to eq. (33);

(D) replace $\varphi_0(y)$ by $\varphi^*(y)$ as the new initial guess, i.e., $\varphi_0(y) = \varphi^*(y)$.

In the nth times of iteration, write

$$\hat{\Phi}_n(y) = \varphi^*(y), \qquad \hat{\Xi}_{n-1}(y) = S^*(y).$$

Then, the procedures of the first-order HAM iteration approach are expressed by

$$\begin{cases}\hat{\Xi}_{n-1}(y) = \dfrac{1}{2}\displaystyle\int_0^1 \frac{1}{\varepsilon^2}G(y,\varepsilon)\hat{\Phi}^2_{n-1}(\varepsilon)\mathrm{d}\varepsilon,\\ \hat{\Phi}_n(y) = (1-\theta)\hat{\Phi}_{n-1}(y) - \theta\displaystyle\int_0^1 K(y,\varepsilon)Q\mathrm{d}\varepsilon\\ \qquad - \theta\displaystyle\int_0^1 \frac{1}{\varepsilon^2}K(y,\varepsilon)\hat{\Phi}_{n-1}(\varepsilon)\hat{\Xi}_{n-1}(\varepsilon)\mathrm{d}\varepsilon.\end{cases} \quad (34)$$

Since the HAM provides us freedom to choose the initial guess, let us choose the initial guess

$$\hat{\Phi}_0(y) = -\frac{Q\theta}{2}\left[(\lambda+1)y - y^2\right]. \quad (35)$$

Note that, eqs. (34) and (35) are exactly the same as the iterative procedures (9) and (10) of the interpolation iterative method [3]. Thus, the interpolation iterative method [3] is indeed a special case of the first-order HAM iteration approach when $c_1 = -\theta$ and $c_2 = -1$. It should be emphasized that the interpolation iterative method [3] is valid for uniform external pressure to arbitrary magnitude, as proved by Zheng and Zhou [4]. So, according to Zheng and Zhou's convergence theorem [4], the HAM-based approach mentioned in sect. 2 is valid for an uniform pressure to arbitrary amplitude, at least in some special cases such as $c_1 = -\theta$ and $c_2 = -1$. This once again reveals the important role of the convergence-control parameters c_1 and c_2 in the frame of the HAM. Indeed, it is the so-called convergence control parameter that differs the HAM from all other analytic approximation methods.

3 HAM approach for given central deflection

Note that the external uniform pressure Q is used in the interpolation iterative method [3]. For details, please refer to

Zheng [1]. However, according to Chien [19], it makes sense to introduce the central deflection into the Von Kármán's plate equations so as to enlarge the convergent region. Based on this knowledge, we propose here a HAM approach for a given central deflection to solve the Von Kármán's plate equations in integral form with the clamped boundary condition. This new HAM-based approach is even more efficient than the previous one for a given Q described in sect. 2.

3.1 Mathematical formulas

Given

$W(0) = a,$

we have from eq. (7) an additional restriction equation

$$\int_0^1 \frac{1}{\varepsilon}\varphi(\varepsilon)\mathrm{d}\varepsilon = -a. \tag{36}$$

Let $\varphi_0(y)$ and $S_0(y)$ denote initial guesses of $\varphi(y)$ and $S(y)$, which satisfy the restriction condition (36), c_1 and c_2 the convergence-control parameters, $q \in [0,1]$ the embedding parameter, respectively. Note that the external uniform pressure Q is unknown here for a given central deflection a. Then, we construct the so-called zeroth-order deformation equations

$$\begin{aligned}&(1-q)\left[\tilde{\Phi}(y,q)-\varphi_0(y)\right]\\ =&c_1 q\left[\tilde{\Phi}(y,q)+\int_0^1 \frac{1}{\varepsilon^2}K(y,\varepsilon)\tilde{\Phi}(\varepsilon,q)\tilde{\Xi}(\varepsilon,q)\mathrm{d}\varepsilon\right.\\ &\left.+\int_0^1 K(y,\varepsilon)\tilde{\Theta}(q)\mathrm{d}\varepsilon\right],\end{aligned} \tag{37}$$

$$\begin{aligned}&(1-q)\left[\tilde{\Xi}(y,q)-S_0(y)\right]\\ =&c_2 q\left[\tilde{\Xi}(y,q)-\frac{1}{2}\int_0^1 \frac{1}{\varepsilon^2}G(y,\varepsilon)\tilde{\Phi}^2(\varepsilon,q)\mathrm{d}\varepsilon\right],\end{aligned} \tag{38}$$

subject to the restriction condition

$$\int_0^1 \frac{1}{\varepsilon}\tilde{\Phi}(\varepsilon,q)\mathrm{d}\varepsilon = -a. \tag{39}$$

Note that $\tilde{\Phi}(y,q), \tilde{\Xi}(y,q)$ and $\tilde{\Theta}(q)$ correspond to the unknown $\varphi(y), S(y)$ and Q, respectively, as mentioned below. Similar to sect. 2.1, we can expand $\tilde{\Phi}(y,q)$, $\tilde{\Xi}(y,q)$ and $\tilde{\Theta}(q)$ into the Maclaurin series:

$$\begin{cases}\tilde{\Phi}(y,q) = \sum\limits_{k=0}^{+\infty}\varphi_k(y)q^k,\\ \tilde{\Xi}(y,q) = \sum\limits_{k=0}^{+\infty}S_k(y)q^k,\\ \tilde{\Theta}(q) = \sum\limits_{k=0}^{+\infty}Q_k\, q^k,\end{cases} \tag{40}$$

in which

$$\begin{cases}\varphi_k(y) = \mathcal{D}_k\left[\tilde{\Phi}(y,q)\right],\\ S_k(y) = \mathcal{D}_k\left[\tilde{\Xi}(y,q)\right],\\ Q_k = \mathcal{D}_k\left[\tilde{\Theta}(q)\right],\end{cases} \tag{41}$$

with the definition $\mathcal{D}_k$ by eq. (18). The so-called homotopy-series solutions read

$$\varphi(y) = \sum_{k=0}^{+\infty}\varphi_k(y), \quad S(y) = \sum_{k=0}^{+\infty}S_k(y), \quad Q = \sum_{k=0}^{+\infty}Q_k. \tag{42}$$

Substituting the Maclaurin series (40) into the zeroth-order deformation eqs. (37)-(39), and then equating the like-power of q, we have the so-called kth-order deformation equations

$$\begin{aligned}&\varphi_k(y)-\chi_k\varphi_{k-1}(y)\\ =&c_1\left[\varphi_{k-1}(y)+\int_0^1 \frac{1}{\varepsilon^2}K(y,\varepsilon)\sum_{i=0}^{k-1}\varphi_i(\varepsilon)S_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\right.\\ &\left.+\int_0^1 K(y,\varepsilon)Q_{k-1}\mathrm{d}\varepsilon\right],\end{aligned} \tag{43}$$

$$\begin{aligned}&S_k(y)-\chi_k S_{k-1}(y)\\ =&c_2\left[S_{k-1}(y)-\int_0^1 \frac{1}{2\varepsilon^2}G(y,\varepsilon)\sum_{i=0}^{k-1}\varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\right],\end{aligned} \tag{44}$$

subject to the restriction condition

$$\int_0^1 \frac{1}{\varepsilon}\varphi_k(\varepsilon)\mathrm{d}\varepsilon = 0, \tag{45}$$

where χ_k is defined by eq. (24), and Q_{k-1} can be determined by the restriction condition (45). Consequently, $\varphi_k(y)$ and $S_k(y)$ of eqs. (43)-(45) are obtained. Then, we have the nth-order approximation:

$$\varphi(y) = \sum_{k=0}^{n}\varphi_k(y), \quad S(y) = \sum_{k=0}^{n}S_k(y), \quad Q = \sum_{k=0}^{n}Q_k. \tag{46}$$

We choose

$$\varphi_0(y) = \frac{-2a}{2\lambda+1}\left[(\lambda+1)y-y^2\right], \qquad S_0(y) = 0, \tag{47}$$

as the initial guesses of $\varphi(y)$ and $S(y)$, respectively. Besides, we also set

$$c_1 = c_2 = c_0, \tag{48}$$

so as to simplify the choice of the optimal convergence-control parameters.

3.2 Results given by the non-iteration HAM approach

Without loss of generality, let us consider the case of $a = 5$, corresponds to $w(0)/h = 3.0$. Note that the convergence-control parameter c_0 is unknown at the beginning. Its optimal value (i.e., −0.25 in this case) is determined by the minimum of the squared residual error $\mathcal{E}$ defined by eq. (28). According to Table 4, the squared residual error quickly decreases to 4×10^{-7} by means of $c_0 = -0.25$. Note that, the Chien's perturbation method [19] (using a as the perturbation quantity) is only valid for $w(0)/h < 2.44$, equivalent to $a < 4$, for a circular plate with clamped boundary. This again illustrates the superiority of the HAM-based approach over the perturbation method [19].

Given a value of a, the optimal convergence-control parameter c_0 can be obtained in a similar way, which can be expressed by such an empirical formula:

$$c_0 = -\frac{11}{11 + a^2}, \qquad (0 < a \le 5). \tag{49}$$

The convergent results of the external uniform pressure Q in case of different values of a are given in Table 5.

3.3 Convergence acceleration by means of iteration

In this subsection, the first-order HAM iteration approach is used for a given central deflection. Without loss of generality, let us first consider the same case of $a = 5$. Note that the squared residual error quickly decreases to 5×10^{-23}, as shown in Table 6. In addition, as shown in Figure 2, it is much faster to obtain convergent results by introducing $W(0)$ into the Von Kármán's plate equations. Therefore, using the central deflection indeed makes sense. It should be emphasized that the HAM-based iteration approach converges much faster than the interpolation iterative method [3], as shown in Figure 2.

Similarly, the convergent results can be obtained by means of the optimal convergence-control parameter with the empirical formula:

$$c_0 = -\frac{25}{25 + a^2}, \tag{50}$$

within the range of $a \le 35$, equivalent to $w(0)/h = 21.2$, as shown in Table 7. According to Figure 3, as a increases, the

Table 4 The squared residual error $\mathcal{E}$ and the approximations of Q in the case of $a = 5$ for a circular plate with clamped boundary, given by the HAM without iteration using $c_0 = -0.25$

k, order of approx.	$\mathcal{E}$	Q
20	3×10^{-2}	132.3
40	2×10^{-3}	132.5
60	9×10^{-5}	132.3
80	1×10^{-6}	132.2
100	4×10^{-7}	132.2

Table 5 The results of the uniform pressure Q in case of different values of a for a circular plate with clamped boundary, given by the HAM approach without iteration using the optimal c_0 given by eq. (49)

a	c_0	Q
1	−0.92	4.8
2	−0.73	14.6
3	−0.55	35.2
4	−0.41	72.4
5	−0.31	132.2

Table 6 The squared residual error $\mathcal{E}$ and the approximations of Q versus the iteration times in the case of $a = 5$ for a circular plate with clamped boundary, given by the first-order HAM iteration approach using $c_0 = -0.5$

m, times of iteration.	$\mathcal{E}$	Q
10	7×10^{-5}	132.2
20	2×10^{-10}	132.2
30	1×10^{-18}	132.2
40	5×10^{-23}	132.2

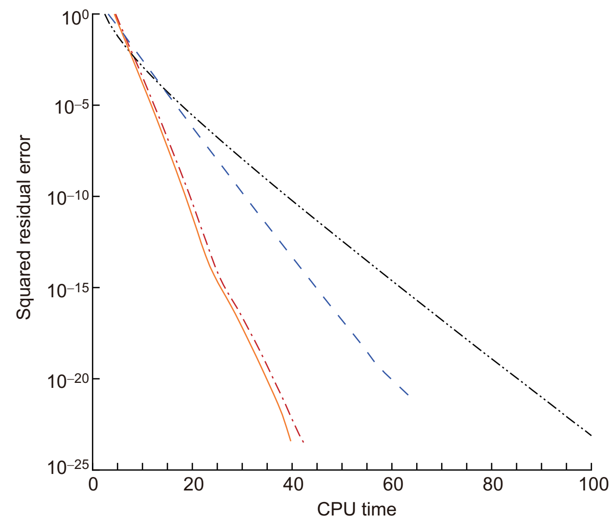

Figure 2 (Color online) The squared residual error $\mathcal{E}$ versus the CPU time in the case of $Q = 1000$ (corresponding to $w(0)/h = 6.1$), given by the interpolation iterative method [3], and the HAM-based approach for given external uniform pressure Q and central deflection a, respectively. Dash-double-dotted line: results given by the interpolation iterative method [3] using the interpolation parameter $\theta = 0.02$; dashed line: results given by the first-order HAM iteration approach for the Von Karman's plate equations in integral form with a given external uniform pressure Q using $c_0 = -0.05$; dash-dotted line: results given by the first-order HAM iteration approach [5] for the Von Kármán's plate equations in differential form with a given central deflection a using $c_0 = -0.25$; solid line: results given by the first-order HAM iteration approach for the Von Kármán's plate equations in integral form with a given central deflection a using $c_0 = -0.25$.

Table 7 The approximations of Q in case of different values of a for a circular plate with clamped boundary, given by the first-order HAM iteration approach with the optimal convergence-control parameter c_0 given by eq. (50)

a	c_0	Q
5	−0.50	132.2
15	−0.10	3152.1
25	−0.04	14334.1
35	−0.02	39053.6

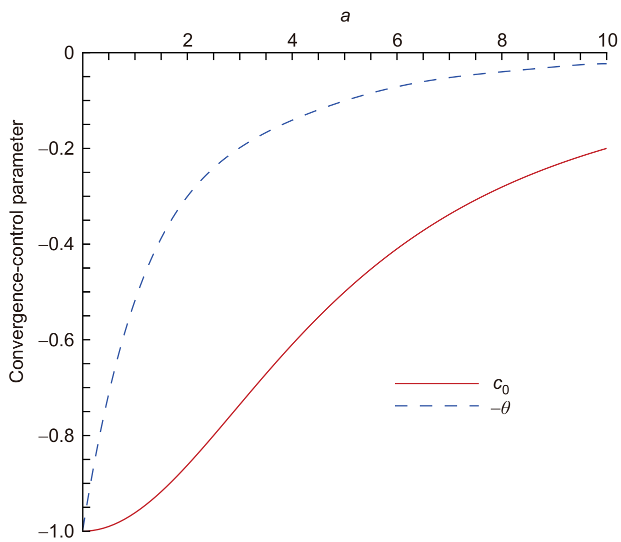

Figure 3 (Color online) The convergence-control parameter c_0 and the interpolation parameter θ versus the central deflection a for a circular plate with clamped boundary, given by the first-order HAM iteration approach for given central deflection a and the interpolation iterative method [1] for given external uniform pressure Q. Dashed line: $-\theta$; solid line: c_0 given by eq. (50).

interpolation iterative parameter θ in the interpolation iterative method [3] tends to 0 much faster than the optimal convergence-control parameter c_0 given by eq. (50). This explains why the HAM-based iteration approach converges much faster than the interpolation iterative method [3]. This reveals once again the importance of the so-called convergence-control parameters in the frame of the HAM.

3.4 Relations to the HAM for the Von Kármán's plate equations in differential form

Here we prove that the HAM approach [5] for the Von Kármán's plate equations in differential form is only a special case of the HAM approach mentioned in sect. 3.1.

Rewriting eqs. (20) and (21), we have

$$\begin{aligned}&\varphi_k(y)\\ =&\chi_k\varphi_{k-1}(y)+c_1\left[\varphi_{k-1}(y)+Q_{k-1}\left(-\frac{y^2}{2}+\frac{\lambda+1}{2}y\right)\right.\\ &+\int_0^y\frac{1}{\varepsilon}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)S_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\\ &+\int_y^1\frac{y}{\varepsilon^2}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)S_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\\ &\left.+(\lambda-1)\int_0^1\frac{y}{\varepsilon}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)S_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\right],\end{aligned}\tag{51}$$

$$\begin{aligned}&S_k(y)\\ =&\chi_kS_{k-1}(y)+c_2\left[S_{k-1}(y)\right.\\ &-\frac{\mu-1}{2}\int_0^1\frac{y}{\varepsilon}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\\ &-\frac{1}{2}\int_0^y\frac{1}{\varepsilon}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\\ &\left.-\frac{1}{2}\int_y^1\frac{y}{\varepsilon^2}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\right].\end{aligned}\tag{52}$$

Differentiating eqs. (51) and (52) with respect to y, we gain

$$\begin{aligned}&\frac{\mathrm{d}\varphi_k(y)}{\mathrm{d}y}\\ =&(c_1+\chi_k)\frac{\mathrm{d}\varphi_{k-1}(y)}{\mathrm{d}y}\\ &+c_1\left[(\lambda-1)\int_0^1\frac{1}{\varepsilon}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)S_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\right.\\ &\left.+\int_y^1\frac{1}{\varepsilon^2}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)S_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon+Q_{k-1}\left(\frac{\lambda+1}{2}-y\right)\right],\end{aligned}\tag{53}$$

$$\begin{aligned}&\frac{\mathrm{d}S_k(y)}{\mathrm{d}y}\\ =&(c_2+\chi_k)\frac{\mathrm{d}S_{k-1}(y)}{\mathrm{d}y}\\ &-\frac{c_2}{2}\left[(\mu-1)\int_0^1\frac{1}{\varepsilon}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\right.\\ &\left.+\int_y^1\frac{1}{\varepsilon^2}\sum_{i=0}^{k-1}\varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon)\mathrm{d}\varepsilon\right],\end{aligned}\tag{54}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2\varphi_k(y)}{\mathrm{d}y^2}=&(c_1+\chi_k)\frac{\mathrm{d}^2\varphi_{k-1}(y)}{\mathrm{d}y^2}\\ &-c_1\left[\frac{1}{y^2}\sum_{i=0}^{k-1}\varphi_i(y)S_{k-1-i}(y)+Q_{k-1}\right],\end{aligned}\tag{55}$$

$$\begin{aligned}\frac{\mathrm{d}^2S_k(y)}{\mathrm{d}y^2}=&(c_2+\chi_k)\frac{\mathrm{d}^2S_{k-1}(y)}{\mathrm{d}y^2}\\ &+c_2\frac{1}{2y^2}\sum_{i=0}^{k-1}\varphi_i(y)\varphi_{k-1-i}(y).\end{aligned}\tag{56}$$

Setting $y = 0$ in eqs. (51) and (52), it holds

$$\begin{cases} \varphi_k(0) = (c_1 + \chi_k)\varphi_{k-1}(0), \\ S_k(0) = (c_2 + \chi_k)S_{k-1}(0), \end{cases} \tag{57}$$

which lead to

$$\begin{cases} \varphi_k(0) = (c_1 + \chi_k)\varphi_{k-1}(0) = \cdots = c_1(c_1+1)^{k-1}\varphi_0(0), \\ S_k(0) = (c_2 + \chi_k)S_{k-1}(0) = \cdots = c_2(c_2+1)^{k-1}S_0(0). \end{cases} \tag{58}$$

Similarly, setting $y = 1$ in eqs. (51)-(54), we have

$$\begin{aligned} &\varphi_k(1) - (c_1 + \chi_k)\varphi_{k-1}(1) \\ =&\lambda\, c_1 \left[\int_0^1 \frac{1}{\varepsilon} \sum_{i=0}^{k-1} \varphi_i(\varepsilon) S_{k-1-i}(\varepsilon) \mathrm{d}\varepsilon + \frac{Q_{k-1}}{2} \right], \end{aligned} \tag{59}$$

$$\begin{aligned} &\left.\frac{\mathrm{d}\varphi_k(y)}{\mathrm{d}y}\right|_{y=1} - (c_1 + \chi_k) \left.\frac{\mathrm{d}\varphi_{k-1}(y)}{\mathrm{d}y}\right|_{y=1} \\ =&(\lambda - 1)c_1 \left[\int_0^1 \frac{1}{\varepsilon} \sum_{i=0}^{k-1} \varphi_i(\varepsilon) S_{k-1-i}(\varepsilon) \mathrm{d}\varepsilon + \frac{Q_{k-1}}{2} \right], \end{aligned} \tag{60}$$

$$\begin{aligned} &S_k(1) - (c_2 + \chi_k) S_{k-1}(1) \\ =& -\frac{\mu\, c_2}{2} \int_0^1 \frac{1}{\varepsilon} \sum_{i=0}^{k-1} \varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon) \mathrm{d}\varepsilon, \end{aligned} \tag{61}$$

$$\begin{aligned} &\left.\frac{\mathrm{d}S_k(y)}{\mathrm{d}y}\right|_{y=1} - (c_2 + \chi_k) \left.\frac{\mathrm{d}S_{k-1}(y)}{\mathrm{d}y}\right|_{y=1} \\ =& -\frac{(\mu - 1)c_2}{2} \int_0^1 \frac{1}{\varepsilon} \sum_{i=0}^{k-1} \varphi_i(\varepsilon)\varphi_{k-1-i}(\varepsilon) \mathrm{d}\varepsilon. \end{aligned} \tag{62}$$

Then, it is straight-forward to gain

$$\begin{aligned} &\left.\left[\varphi_k(y) - \frac{\lambda}{\lambda - 1} \frac{\mathrm{d}\varphi_k(y)}{\mathrm{d}y} \right]\right|_{y=1} \\ =&(c_1 + \chi_k) \left.\left[\varphi_{k-1}(y) - \frac{\lambda}{\lambda - 1} \frac{\mathrm{d}\varphi_{k-1}(y)}{\mathrm{d}y} \right]\right|_{y=1} \\ =&\cdots \\ =&c_1(c_1 + 1)^{k-1} \left.\left[\varphi_0(y) - \frac{\lambda}{\lambda - 1} \frac{\mathrm{d}\varphi_0(y)}{\mathrm{d}y} \right]\right|_{y=1}, \end{aligned} \tag{63}$$

and

$$\begin{aligned} &\left.\left[S_k(y) - \frac{\mu}{\mu - 1} \frac{\mathrm{d}S_k(y)}{\mathrm{d}y} \right]\right|_{y=1} \\ =&(c_2 + \chi_k) \left.\left[S_{k-1}(y) - \frac{\mu}{\mu - 1} \frac{\mathrm{d}S_{k-1}(y)}{\mathrm{d}y} \right]\right|_{y=1} \\ =&c_2(c_2 + 1)^{k-1} \left.\left[S_0(y) - \frac{\mu}{\mu - 1} \frac{\mathrm{d}S_0(y)}{\mathrm{d}y} \right]\right|_{y=1}. \end{aligned} \tag{64}$$

If we choose $c_1 = c_2 = c_0$ and let the initial guesses $\varphi_0(y)$ and $S_0(y)$ be the same as that chosen in the HAM approach [5], which satisfy the boundary conditions of the Von Kármán's plate equations in differential form, i.e.,

$$\begin{cases} \varphi_0(0) = S_0(0) = 0, \qquad \varphi_0(1) = \dfrac{\lambda}{\lambda - 1} \left.\dfrac{\mathrm{d}\varphi_0(y)}{\mathrm{d}y}\right|_{y=1}, \\ S_0(1) = \dfrac{\mu}{\mu - 1} \left.\dfrac{\mathrm{d}S_0(y)}{\mathrm{d}y}\right|_{y=1}, \qquad \displaystyle\int_0^1 \frac{1}{\varepsilon}\varphi_0(\varepsilon)\mathrm{d}\varepsilon = -a, \end{cases} \tag{65}$$

then according to eqs. (58), (63) and (64), we have

$$\begin{cases} \varphi_k(0) = S_k(0) = 0, \qquad \varphi_k(1) = \dfrac{\lambda}{\lambda - 1} \left.\dfrac{\mathrm{d}\varphi_k(y)}{\mathrm{d}y}\right|_{y=1}, \\ S_k(1) = \dfrac{\mu}{\mu - 1} \left.\dfrac{\mathrm{d}S_k(y)}{\mathrm{d}y}\right|_{y=1}. \end{cases} \tag{66}$$

Note that, the governing eqs. (55) and (56), the restriction condition (45), and the boundary conditions (66) are exactly the same as the HAM approach [5] for the Von Kármán's plate equations in differential form (for details, please refer to sect. 2 in ref. [5]). So, the HAM approach [5] for the Von Kármán's plate equations in differential form is just a special case of the HAM mentioned above in sect. 3.1.

Note that, if we choose $c_1 \in (-2, 0)$ and $c_2 \in (-2, 0)$, then according to eqs. (58), (63) and (64)), we have

$$\begin{aligned} \varphi(0) =&\varphi_0(0) + \sum_{k=1}^{+\infty} \varphi_k(0) \\ =&\varphi_0(0) \left[1 + \sum_{k=1}^{+\infty} c_1(c_1 + 1)^{k-1} \right] = 0, \end{aligned} \tag{67}$$

$$\begin{aligned} S(0) =&S_0(0) + \sum_{k=1}^{+\infty} S_k(0) \\ =&S_0(0) \left[1 + \sum_{k=1}^{+\infty} c_2(c_2 + 1)^{k-1} \right] = 0, \end{aligned} \tag{68}$$

$$\begin{aligned} &\left.\left[\varphi(y) - \frac{\lambda}{\lambda - 1} \frac{\mathrm{d}\varphi(y)}{\mathrm{d}y} \right]\right|_{y=1} \\ =& \sum_{k=0}^{+\infty} \left.\left[\varphi_k(y) - \frac{\lambda}{\lambda - 1} \frac{\mathrm{d}\varphi_k(y)}{\mathrm{d}} \right]\right|_{y=1} \\ =& \left[1 + \sum_{k=1}^{+\infty} c_1(c_1 + 1)^{k-1} \right] \left.\left[\varphi_0(y) - \frac{\lambda}{\lambda - 1} \frac{\mathrm{d}\varphi_0(y)}{\mathrm{d}y} \right]\right|_{y=1} \\ =&0, \end{aligned} \tag{69}$$

$$\begin{aligned} &\left.\left[S(y) - \frac{\mu}{\mu - 1} \frac{\mathrm{d}S(y)}{\mathrm{d}y} \right]\right|_{y=1} \\ =& \sum_{k=0}^{+\infty} \left.\left[S_k(y) - \frac{\mu}{\mu - 1} \frac{\mathrm{d}S_k(y)}{\mathrm{d}y} \right]\right|_{y=1} \end{aligned}$$

$$=\left[1+\sum_{k=1}^{+\infty}c_2(c_2+1)^{k-1}\right]\left[S_0(y)-\frac{\mu}{\mu-1}\frac{\mathrm{d}S_0(y)}{\mathrm{d}y}\right]\Bigg|_{y=1}$$
$$=0, \qquad (70)$$

since

$$1+\sum_{k=1}^{+\infty}c_n(c_n+1)^{k-1}=1+\frac{c_n}{1-(c_n+1)}=0$$

when $|1+c_n|<1$, i.e., $c_n\in(-2,0)$, for $n=1,2$.

Thus, the homotopy-series solution $\varphi(y)$ and $S(y)$ automatically satisfy the boundary conditions of the Von Kármán's plate equations in integral form as long as $c_1\in(-2,0)$ and $c_2\in(-2,0)$, no matter whatever initial guesses we choose, say, even if they do not satisfy the boundary conditions. In other words, the HAM approach for the Von Kármán's plate equations in integral form mentioned in sect. 3.1 is more general than the HAM approach [5].

4 Concluding remarks

In this paper, the homotopy analysis method (HAM) is successfully applied to solve the Von Kármán's plate equations in integral form for a circular plate with the clamped boundary. Two HAM-based analytic approaches are proposed for either a given external uniform pressure Q or a given central deflection of the plate. Both of them are valid for external uniform pressure to arbitrary magnitude by means of choosing proper values of the so-called convergence-control parameters c_1 and c_2 in the frame of the HAM. Besides, it is found that iteration can greatly accelerate the convergence of solution series. Furthermore, these two HAM-based iteration approaches converge much faster than the interpolation iterative method [3], as shown in Figure 2.

In our previous paper [5] about the Von Kármán's plate equations in differential form, we proved that the well-known Vincent's [18] perturbation method, Chien's [19] perturbation method and the modified iteration method [20] are only the special cases of the HAM when $c_0=-1$. In this paper, using the Von Kármán's plate equations in integral form, we further prove that the interpolation iterative method [3] is also a special case of the first-order HAM iteration approach for a given external uniform pressure Q when $c_1=-\theta$ and $c_2=-1$, where θ denotes the interpolation iterative parameter. Therefore, according to Zheng and Zhou's [4] convergence theorem about the interpolation iterative method [3], the HAM-based approach for the Von Kármán's plate equations in integral form is valid for external uniform pressure to arbitrary magnitude at least in the special cases $c_1=-\theta$ and $c_2=-1$. More importantly, we prove that the HAM [5] for the Von Kármán's plate equations in differential form is also a special case of the HAM for the Von Kármán's plate equations in integral form when we choose the same initial guesses as in ref. [5]. In other word, the HAM-based approaches for the Von Kármán's plate equations in both of the differential and integral forms are equivalent in some cases. Thus, now, it is easy to understand why the HAM-based approach [5] for the Von Kármán's plate equations in differential form is valid for uniform pressure to arbitrary amplitude. All of these illustrate the importance of the so-called convergence-control parameters in the frame of the HAM, and besides reveal the reason why the HAM can guarantee the convergence of solution series for highly nonlinear problems.

Note that the interpolation iterative method [3] is only for given external uniform pressure Q. It is found that, by means of choosing an optimal convergence-control parameter, our HAM-based iterative approach for a given external uniform pressure Q can converge faster than the interpolation iterative method [3]. It is interesting that our HAM-based approach for a given central deflection of the plate is even more efficient than the HAM-based approach for a given external uniform pressure Q. Besides, as shown in Figure 2, the HAM-based approach for the Von Kármán's plate equations in integral form generally converges faster than the HAM-based approach for the Von Kármán's plate equations in differential form. Note that, Unlike the HAM-based approach for the Von Kármán's plate equations in differential form, there is no restrictions on boundary conditions for the initial guesses of the HAM-based approach for the Von Kármán's plate equations in integral form, so that we have greater freedom to choose a better initial guesses.

In summary, in the frame of the HAM, we can derive all previous analytic approximation methods for the famous Von Kármán's plate equations. More importantly, by choosing an optimal value of the so-called convergence-control parameter, all of the HAM-based approaches can efficiently give convergent results for uniform external pressure to arbitrary amplitude, even if the traditional methods become invalid. All of these illustrate the originality, flexibility and potential of the HAM for the famous Von Kármán's plate equations, and show the superiority of the HAM over the perturbation methods. Without doubt, the HAM can be applied to solve some other challenging problems with high nonlinearity.

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