Transport in Porous Media (2005) 61: 365–379
DOI 10.1007/s11242-005-0546-7

Analytic Series Solution for Unsteady Mixed Convection Boundary Layer Flow Near the Stagnation Point on a Vertical Surface in a Porous Medium

JUN CHENG[1], SHIJUN LIAO[1,*] and IOAN POP[2]
[1]*School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200030, P.R. China*
[2]*Faculty of Mathematics, University of Cluj, R-3400 Cluj CP253, Romania*

(Received: 26 October 2004; accepted in final from: 11 January 2005)

Abstract. In this paper, we solve the unsteady mixed convection flow near the stagnation point on a heated vertical flat plate embedded in a Darcian fluid-saturated porous medium by means of an analytic technique, namely the Homotopy Analysis Method. Different from previous perturbation results, our analytic series solutions are accurate and uniformly valid for all dimensionless times and for all possible values of mixed convection parameter, and besides agree well with numerical results. This provides us with a new analytic approach to investigate related unsteady problems.

Key words: unsteady mixed convection, boundary layer flows, stagnation point, homotopy analysis method.

1. Introduction

The investigation of convective heat transfer in fluid-saturated porous media has many important applications in technology and geothermal energy recovery, such as oil recovery, food processing, fiber and granular insulation, design of packed bed reactors, dispersion of chemical contaminants in various processes in the chemical industry and in the environment, etc. Contributions have been made by many researchers, such as Ingham and Pop (1998), Nield and Bejan (1999), Vafai (2000), Pop and lngham (2001), Bejan and Kraus (2003), Ingham *et al.* (2004), Bejan *et al.* (2004), Johnson and Cheng (1978), Merkin (1980), Harris *et al.* (1999), Magyari *et al.* (2004), and Aly *et al.* (2003).

Consider a mixed convection flow at the two-dimensional stagnation point on a double-infinite vertical flat plate embedded in a fluid-saturated

*Author for correspondence: e-mail: sjliao@sjtu.edu.cn

porous medium of constant ambient temperature T_∞. Assume that the external flow starts impulsively in motion from rest towards the plate with a steady velocity $u_e(x)$ at time $t=0$. The unsteady boundary layer equations governing this mixed convection flow are given by Pop and Ingham (2001) as follows:

$$u_x + v_y = 0, \tag{1}$$

$$u = u_e(x) + \frac{gK\beta}{\nu}(T - T_\infty), \tag{2}$$

$$\sigma\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

subject to the initial conditions and boundary conditions

$$t < 0 : u(x, y) = v(x, y) = 0,\ \ T(x, y) = T_\infty \ \text{ for any } x, y, \tag{4}$$

$$t \geq 0 : v(x, 0) = 0, \quad T(x, 0) = T_w - T_\infty, \quad x \geq 0, \tag{5}$$

$$u(x, \infty) = u_e(x) = (U_e/L)x, \quad x \geq 0, \tag{6}$$

where (x, y) denotes the Cartesian coordinates in vertical and horizontal directions, respectively, with the positive y-axis pointing towards the porous medium (external flow), t is the time, u and v are the velocity components along x- and y-axes, T denotes the fluid temperature, L a characteristic length, U_e a characteristic velocity, T_w is the temperature of the plate which is assumed to vary linearly with the distance x along the plate, i.e.

$$T_w = T_\infty + sT_0(x/L),$$

$T_0 > 0$ is a characteristic temperature, g the gravitational acceleration, K the permeability of the porous medium, α_m the effective thermal diffusivity of the porous medium, β the thermal expansion coefficient, ν the kinematic viscosity, σ the ratio of composite material heat capacity to convective fluid heat capacity, and $s = \pm 1$. Note that $s = +1$ corresponds to buoyancy assisting flow and $s = -1$ corresponds to buoyancy opposing flow.

Recently, Nazar *et al*. (2004) gave numerical solutions for the unsteady boundary layer problem by means of the Keller-Box method (1984), and besides gained perturbation solutions for *small* times. For the steady-state flows, they reported perturbation solutions for *small* and *large* values of the mixed convection parameter. However, none of their analytic solutions are valid for all values of the mixed convection parameter.

Liao (2003) developed an analytic method for strongly nonlinear problems, namely the homotopy analysis method (HAM), which has been successfully applied to many nonlinear problems in science and engineering (for example, see Arjab *et al*., 2003; Liao, 2003a, b, c; Liao and Cheung, 2003; Hayat *et al.*, 2004; Liao, 2004). The HAM is based on a traditional concept of homotopy in topology. However, in the frame of the HAM, the concept of homotopy is generalized by means of introducing an auxiliary parameter and an auxiliary function. One first connects selected initial guesses and unknown solutions of a nonlinear problem by constructing such a generalized homotopy with respect to an embedding parameter $q \in [0, 1]$. Then, the solution can be expressed by a kind of Taylor series with respect to the embedding parameter q at $q=0$, and each term of the solution series is governed by a linear equation. In this way, a nonlinear problem is transformed into an infinite number of linear problems. However, different from perturbation techniques, such kind of transformation does not depend on any small/large parameters at all. Besides, different from all previous analytic techniques, the HAM provides us with a simple way to control and adjust the convergence of the solution series, and also the great freedom to choose a proper set of base functions. Furthermore, it logically contains other nonperturbation techniques such as Lyapunov's small parameter method (Lyapunov, 1892), the δ-expansion method (Karmishin *et al.*, 1990), and Adomian's decomposition method (Adomina, 1976), as proved by Liao (2003a). So, the HAM is rather general. Currently, Liao (in press) successfully applied the HAM to the unsteady boundary-layer flows caused by an impulsively stretching plate and obtained analytic series solutions valid for all times $0 \le t < +\infty$. In this paper, we further apply the HAM to give analytic series solutions of the considered problem, which are valid and accurate for all times and all values of the mixed convection parameter.

2. Mathematical Description

Following Williams and Rhyne (1980), one introduces the similarity variables:

$$\eta = (U_e/L\alpha_{\mathrm{m}})^{1/2} y \xi^{-1/2}, \quad \xi = 1 - \exp(-\tau), \quad \tau = (U_e/L\sigma)t, \tag{7}$$

$$u(x, y, t) = (U_e x/L) f'(\eta, \xi), \tag{8}$$

$$v(x, y, t) = -(U_e \alpha_{\mathrm{m}}/L)^{1/2} \xi^{1/2} f(\eta, \xi), \tag{9}$$

$$T(x, y, t) = T_\infty + sT_0(x/L)\theta(\eta, \xi). \tag{10}$$

Using above transformations, Equations (1)–(3) become

$$f' = 1 + \lambda\theta, \tag{11}$$

$$\theta'' + \frac{1}{2}\eta(1-\xi)\theta' + \xi(f\theta' - f'\theta) = \xi(1-\xi)\frac{\partial\theta}{\partial\xi} \tag{12}$$

for $0 \le \xi \le 1$, and the corresponding boundary conditions (4) to (6) read

$$f(0,\xi)=0, \quad \theta(0,\xi)=1, \quad \theta(\eta,\xi)\to 0 \quad \text{as } \eta\to\infty \tag{13}$$

for $0\le\xi\le 1$, Here the mixed convection parameter λ is defined by

$$\lambda = s\left(\frac{Ra}{Pe}\right) \tag{14}$$

where $Ra = gK\beta T_0 L/(\alpha_{\mathrm{m}}\nu)$ is the Rayleigh number and $Pe = U_e L/(\alpha_{\mathrm{m}})$is the Péclet number. Note that $\lambda > 0$ $(s=+1)$ corresponds to buoyancy assisting flow, and $\lambda < 0$ $(s=-1)$ corresponds to buoyancy opposing flow. Thus, Equations (11) and (12) can be combined to be

$$f''' + \frac{1}{2}(1-\xi)\eta f'' + \xi(ff'' + f' - f'^2) = \xi(1-\xi)\frac{\partial f'}{\partial\xi}. \tag{15}$$

subject to the boundary conditions

$$f(0,\xi)=0, \quad f'(0,\xi)=1+\lambda, \quad f'(\infty,\xi)=1. \tag{16}$$

The skin friction coefficient C_f, and the Nusselt number Nu, are defined by

$$C_f = \frac{2\mu(x/L)}{\rho u_e^2(x)}\frac{\partial u}{\partial y}|_{y=0},$$

$$Nu = -\frac{L}{(T_w - T_\infty)}\frac{\partial T}{\partial y}|_{y=0} \tag{17}$$

Using similarity variables defined by (7) to (10), we get

$$C_f/(Pr/Pe)^{1/2} = 2\xi^{-1/2}f''(0,\xi), \tag{18}$$

$$Nu/(PrRe)^{1/2} = -\xi^{-1/2}\theta'(0,\xi) = -\lambda^{-1}\xi^{-1/2}f''(0,\xi), \tag{19}$$

where $Pr = \nu/\alpha_{\mathrm{m}}$ and $Re = U_e L/\nu$ are Prandtl and Reynolds numbers, respectively.

3. Known Solutions

3.1. INITIAL FLOWS

When $\xi=0$, corresponding to $\tau=0$, Equation (15) becomes the Rayleigh-type equation

$$F''' + \frac{1}{2}\eta F'' = 0 \tag{20}$$

subject to the boundary conditions

$$F(0)=0, \quad F'(0)=1+\lambda, \quad F'(\infty)=1, \tag{21}$$

where $F(\eta) = f(\eta,0)$. The above equations has the exact solution

$$F(\eta) = \eta + \lambda \left[\eta \ \mathrm{erfc}(\eta/2) - \frac{2}{\sqrt{\pi}} \exp(-\eta^2/4) + \frac{2}{\sqrt{\pi}} \right], \tag{22}$$

where erfc(z) is the complementary error function defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} \exp(-z^2) \mathrm{d}z. \tag{23}$$

Thus, the corresponding local skin friction coefficient and the Nusselt number in the initial unsteady flow are

$$C_f/(Pr/Re)^{1/2} = -\frac{2\lambda}{\sqrt{\pi}} \xi^{-1/2}, \tag{24}$$

$$Nu/(PrRe)^{1/2} = \frac{1}{\sqrt{\pi}} \xi^{-1/2}. \tag{25}$$

3.2. STEADY-STATE FLOWS

When $\xi = 1$, corresponding to $\tau \to \infty$, Equation (15) becomes

$$G''' + GG'' + G' - G'^2 = 0, \tag{26}$$

subject to the boundary conditions

$$G(0) = 0, \quad G'(0) = 1 + \lambda, \quad G'(\infty) = 1, \tag{27}$$

where $G(\eta) = f(\eta, 1)$. Using perturbation method, Nazar *et al.* (2004) give analytic approximations for small λ, i.e.

$$G''(0) = -1.2533\lambda - 0.4068\lambda^2 + \text{h.o.t.} \quad \text{for } |\lambda| \ll 1, \tag{28}$$

and approximations for large λ, i.e.

$$G''(0) = \lambda^{3/2}(-1 - 0.8160\lambda^{-1} + 0.4605\lambda^{-2} + \text{h.o.t.}) \quad \text{for } |\lambda| \gg 1, \tag{29}$$

respectively. However, neither (28) nor (29) are uniformly valid for *all* values of λ, as shown in Figure 3.

3.3. SOLUTION FOR SMALL ξ AND τ

By means of perturbation technique, it is assumed that a solution of Equations (15) and (16) for small values of ξ ($\ll 1$) is of the form

$$f(\eta, \xi) = f_0(\eta) + f_1(\eta)\xi + f_2(\eta)\xi^2 + \text{h.o.t.} \tag{30}$$

Nazar *et al.* (2004) give the local Nusselt number

$$\begin{aligned} Nu/(PrRe)^{1/2} &= -\lambda^{-1}\xi^{-1/2} f''(0, \xi) \\ &= \frac{1}{\sqrt{\pi}} \left[\xi^{-1/2} - \left(\frac{4\lambda}{3\pi} - \frac{5 + 6\lambda}{4} \right) \xi^{1/2} + \ \text{h.o.t.} \right], \end{aligned} \tag{31}$$

for small ξ and τ ($\ll 1$) , where

$$\xi = \tau - \frac{1}{2}\tau^2 + \frac{1}{6}\tau^3 + \text{h.o.t.} \tag{32}$$

4. Homotopy Analysis Method

In this section we employ the HAM to give analytic solutions of Equations (15) and (16), which are uniformly valid for all times $0 \le \tau < +\infty$ and all values of the parameter λ. In general, the temperature and velocity of the fluid decays exponentially at infinity, i.e. $f' \to 1$ exponentially (note that a few of flows decay algebraically). So, the solution should contain the term $\exp(-n\eta), n \ge 1$. Besides, ξ and η appear in Equation (15) and (16). So, it is reasonable to assume that $f(\eta, \xi)$ could be expressed by the following set of base functions

$$\{\xi^m \eta^k \exp(-n\eta) \mid k \ge 0, m \ge 0, n \ge 0\} \tag{33}$$

in the form

$$f(\eta, \xi) = a_0^{0,0} + \eta + \sum_{k=0}^{+\infty}\sum_{i=0}^{+\infty}\sum_{j=1}^{+\infty} a_k^{i,j} \xi^k \eta^i \ \exp(-j\eta), \tag{34}$$

where $a_k^{i,j}$ are constant coefficients. The above expression provides us with the so called *Rule of Solution Expression* (see Liao, 2003a), which plays an important role in the frame of the HAM, as shown later.

In the frame of the HAM, one needs to choose a proper guess of the solution as the lowest order of approximation. In general, a guess solution contains a few terms of the solution expression such as (34), and besides should satisfy the boundary conditions, if possible. So, under the *Rule of Solution Expression* (34) and using the boundary conditions (16), it is straightforward to choose the initial guess

$$f_0(\eta, \xi) = -\lambda \exp(-\eta) + \eta + \lambda. \tag{35}$$

Besides, in order to transform the nonlinear problem into an infinite number of linear problems, we need an auxiliary linear operator $\mathcal{L}$. From Equation (15), $\mathcal{L}$ should be a second-order differential operator, i. e.

$$\mathcal{L}\phi = \phi''' + A_2\phi'' + A_1\phi' + A_0\phi,$$

where the prime denotes the differentiation with respect to η, A_0, A_l and A_2 are constants to be determined soon. Clearly, the equation

$$\mathcal{L}\phi = 0$$

has the general solution

$$\phi = C_1\phi_1 + C_2\phi_2 + C_3\phi_3,$$

where C_1, C_2 and C_3 are any constants, which should be determined by the boundary conditions (16). Obviously, ϕ_1, ϕ_2 and ϕ_3 should be chosen under the Rule of Solution Expression (34). As the first attempt, we can choose

$$\phi_1 = 1, \quad \phi_2 = \exp(-\eta), \quad \phi_3 = \exp(-2\eta).$$

However, all of them vanishes at infinity, and thus the boundary condition at infinity is satisfied automatically. Thus, only two constants of C_1, C_2 and C_3 can be determined by the two boundary conditions at $\eta = 0$. To avoid this, we may choose

$$\phi_1 = 1, \quad \phi_2 = \exp(-\eta), \quad \phi_3 = \exp(\eta).$$

To satisfy the boundary condition at infinity, $C_3 = 0$ must hold, and then C_1 and C_2 can be determined by the two boundary conditions at $\eta = 0$. Enforcing

$$\mathcal{L}[1] = 0, \quad \mathcal{L}[\exp(-\eta)] = 0, \quad \mathcal{L}[\exp(\eta)] = 0,$$

we have

$$A_2 = 0, \quad A_1 = -1, \quad A_0 = 0.$$

Thus, we should choose the auxiliary linear operator

$$\mathcal{L}[\Phi(\eta, \xi; q)] = \frac{\partial^3 \Phi}{\partial \eta^3} - \frac{\partial \Phi}{\partial \eta}, \tag{36}$$

where $q \in [0, 1]$ is an embedding parameter. Notice that the auxiliary linear operator $\mathcal{L}$ has the property

$$\mathcal{L}[C_1 + C_2 \ \exp(-\eta) + C_3 \ \exp(\eta)] = 0 \tag{37}$$

for any constant coefficients C_1, C_2, and C_3. From (15), we are led to define a nonlinear operator

$$\mathcal{N}[\Phi(\eta, \xi; q)] = \frac{\partial^3 \Phi}{\partial \eta^3} + \frac{1}{2}(1-\xi)\eta \frac{\partial^2 \Phi}{\partial \eta^2} + \xi \left[\Phi \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{\partial \Phi}{\partial \eta} - \left(\frac{\partial \Phi}{\partial \eta} \right)^2 \right] - \xi(1-\xi) \frac{\partial^2 \Phi}{\partial \eta \partial \xi}. \tag{38}$$

By means of the HAM, we first introduce a nonzero auxiliary parameter $\hbar$ to construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\Phi(\eta, \xi; q) - f_0(\eta, \xi)] = \hbar q \mathcal{N}[\Phi(\eta, \xi; q)], \tag{39}$$

subject to the boundary conditions

$$\Phi(0, \xi; q) = 0, \quad \left.\frac{\partial \Phi(\eta, \xi; q)}{\partial \eta}\right|_{\eta=0} = 1 + \lambda, \left.\frac{\partial \Phi(\eta, \xi; q)}{\partial \eta}\right|_{\eta=+\infty} = 1, \tag{40}$$

where $q \in [0, 1]$ is an embedding parameter. Obviously, when $q = 0$. we have

$$\Phi(\eta, \xi; 0) = f_0(\eta, \xi). \tag{41}$$

When $q = 1$, the zeroth-order deformation equations are equivalent to the original ones (15) and (16), provided

$$\Phi(\eta, \xi; 1) = f(\eta, \xi). \tag{42}$$

Thus, as the embedding parameter q increases from 0 to 1, $\Phi(\eta, \xi; q)$, governed by the zeroth-order deformation Equations (39) and (40), varies (or deforms) from the initial guess $f_0(\eta, \xi)$ to the solution $f(\eta, \xi)$ of the original Equations (15) and (16). Assume that the auxiliary parameter $\hbar$ is so properly chosen that the Taylor series of $\Phi(\eta, \xi; q)$ expanded with respect to the embedding parameter q, i.e.

$$\Phi(\eta, \xi; q) = \Phi(\eta, \xi; 0) + \sum_{n=1}^{+\infty} f_n(\eta, \xi) q^n,$$

converges at $q = 1$, where

$$f_n(\eta, \xi) = \frac{1}{n!} \left.\frac{\partial^n \Phi(\eta, \xi; q)}{\partial q^n}\right|_{q=0}. \tag{43}$$

Then, we have from (41) and (42) that

$$f(\eta, \xi) = f_0(\eta, \xi) + \sum_{n=1}^{+\infty} f_n(\eta, \xi). \tag{44}$$

The governing equations and boundary conditions for the unknown $f_n(\eta, \xi)$ can be deduced from the zeroth-order deformation equation. For simplicity, define the vectors

$$\vec{f}_n = \{f_0, f_1, f_2, \ldots, f_n\}. \tag{45}$$

Differentiating the zeroth-order deformation Equations (39) and (40) m times with respect to the embedding parameter q, then dividing by $m!$, and finally setting $q = 0$, we have the mth-order deformation equations

$$\mathcal{L}[f_m(\eta, \xi) - \chi_m f_{m-1}(\eta, \xi)] = \hbar R_m(\vec{f}_{m-1}, \eta, \xi), \tag{46}$$

subject to the boundary conditions

$$f_m(0,\xi)=0, \quad \left.\frac{\partial f_m(\eta,\xi)}{\partial \eta}\right|_{\eta=0}=0, \quad \left.\frac{\partial f_m(\eta,\xi)}{\partial \eta}\right|_{\eta=+\infty}=0, \tag{47}$$

where

$$\begin{aligned} R_m(\vec{f}_{m-1},\eta,\xi) &= \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathcal{N}[\Phi(\eta,\xi;q)]}{\partial q^{m-1}}\right|_{q=0} \\ &= \frac{\partial^3 f_{m-1}}{\partial \eta^3}+\frac{1}{2}(1-\xi)\eta\frac{\partial^2 f_{m-1}}{\partial \eta^2}-\xi(1-\xi)\frac{\partial^2 f_{m-1}}{\partial\eta\partial\xi}+ \\ &\quad +\xi\frac{\partial f_{m-1}}{\partial \eta}+\xi\sum_{i=0}^{m-1}\left[f_{m-1-i}\frac{\partial^2 f_i}{\partial \eta^2}-\frac{\partial f_{m-1-i}}{\partial \eta}\frac{\partial f_i}{\partial \eta}\right] \end{aligned} \tag{48}$$

and

$$\chi_k=\begin{cases}0, & k\le 1,\\ 1, & k>1.\end{cases} \tag{49}$$

It is easy to solve the above *linear* high-order deformation equations. Let $f_m^*(\eta,\xi)$ denote a special solution of Equation (46). From (37), the corresponding general solution reads

$$f_m(\eta,\xi)=f_m^*(\eta,\xi)+C_1+C_2\exp(-\eta)+C_3\exp(\eta), \tag{50}$$

where the coefficients $C_l, C_2,$ andC_3 are determined by the boundary conditions (47), i.e.

$$C_3=0, \quad C_2=\left.\frac{\partial f_m^*(\eta,\xi)}{\partial \eta}\right|_{\eta=0}, \quad C_1=-C_2-f_m^*(0,\xi). \tag{51}$$

In this way, we can get analytic results at high-order of approximations one after the other in the order $m=1,2,3,\ldots$, by means of a symbolic software such as Mathematica. Note that $f_m(\eta,\xi)$ fits into the Rule of Solution Expression (34).

5. Result and Discussion

Liao (2003a) proved that, as long as a solution series given by the HAM converges, it must be one of the solutions. Thus, it is important to ensure that the solution series (44) converges for all times $0\le\xi\le 1$ and all values of λ. Fortunately, there exists an auxiliary parameter $\hbar$ which can adjust and control the convergence of the solution series, as shown in our previous publications. For simplicity, we use the case of $\xi=0$ as an example. To choose a proper value of $\hbar$, we plot the so-called $\hbar$-curves, as

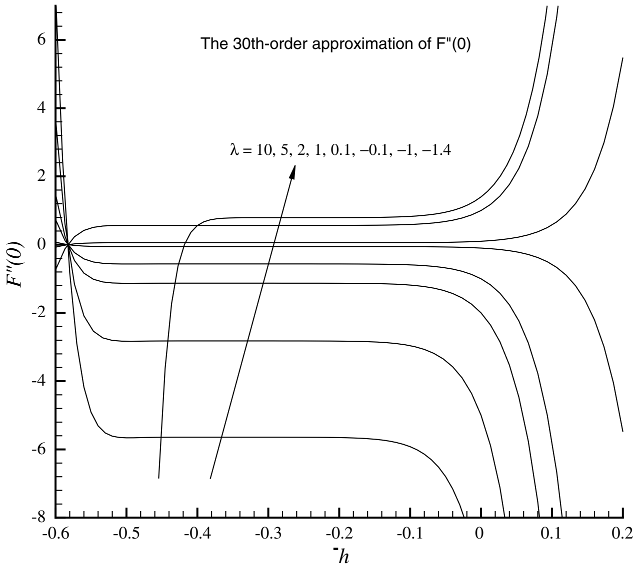

Figure 1. $\hbar$-curves of $F''(0) = f''(0, 0)$ at $\xi = 0$ for different values of λ.

shown in Figure 1, where $\hbar$ is regarded as a variable and $F''(0) = f''(0, 0)$ a function $\hbar$. For given λ, there exists a parallel segment line which corresponds to a region $\hbar$, for example, $-0.5 < \hbar < -0.1$ for $\lambda = 10$, $-0.35 < \hbar < -0.1$ for $\lambda = -1.4$ (note that there is no solution for $\lambda < -1.41175$, see Nazar *et al.*, 2004), and so on. Note that this parallel segment of $\hbar$-independent solutions is identical with convergent series and thus is the unique solution. Any value of $\hbar$ in the parallel region guarantees the convergence of the solution series, and the solutions outside of this segment represent divergent series. As an example, we can have $\hbar = -1/4$ for $-1 \leq \lambda \leq 5$, $\hbar = -1/5$ for $\lambda = -1.4$, and $\hbar = -1/6$ for $\lambda = -10$, respectively. And besides one can employ the so-called homotopy-Padé technique (Liao and Cheung, 2003) to accelerate the convergence. The 30th-order approximation when $\xi = 0$ and $\hbar = -1/5$ and the corresponding [3,3] homotopy-Padé approximation of the velocity profile $f'(\eta, 0)$ agree well with the exact solution (22) in the whole time period $0 \leq \eta < +\infty$, as shown in Figure 2.

Similarly, for any values of given ξ and λ, we can find a proper value of $\hbar$ to ensure that the solution series converges. It is found that if the solution series converges when $\xi = 1$ by means of a chosen value of $\hbar$, the solution series converges for all times $0 \leq \xi \leq 1$ by using the same value of $\hbar$. For example, our solution series converge in the whole time period $0 \leq \xi \leq 1$ for $-0.1 \leq \lambda \leq 5$ by means of $\hbar = -1/4$, for $\lambda = 1.4$ by means of $\hbar = -1/5$, and for $\lambda = 10$ by means of $\hbar = -1/6$, respectively. Figure 3 shows the steady state behavior of $G''(0)$ of Equations (26) and (27) when $\xi = 1$. Different from the perturbation solutions for small $\lambda (\lambda \ll 1)$ or large $\lambda (\lambda \gg 1)$ given by

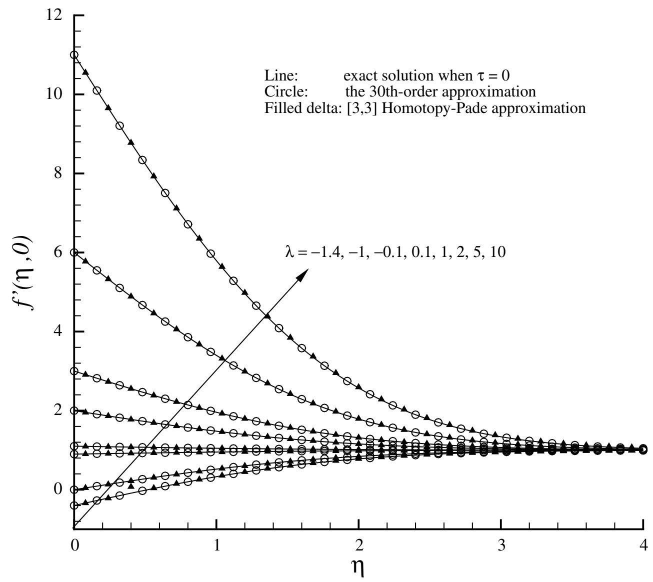

Figure 2. Comparison of $f'(\eta, 0)$ of the exact solution (22) with the 30th-order approximation when $\hbar = -1/5, \xi = 0$, and the [3,3] homotopy-Padé approximation.

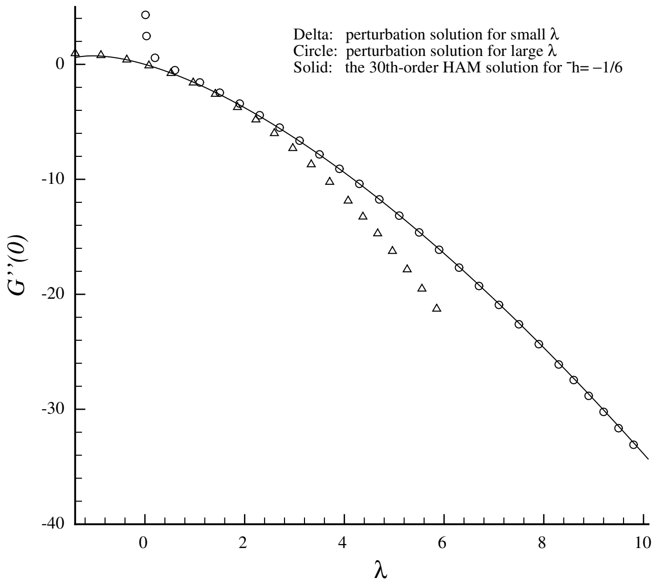

Figure 3. The skin friction $G''(0) = f''(0, 1)$ with respect to λ for the steady-state flow $(\xi = 1)$.

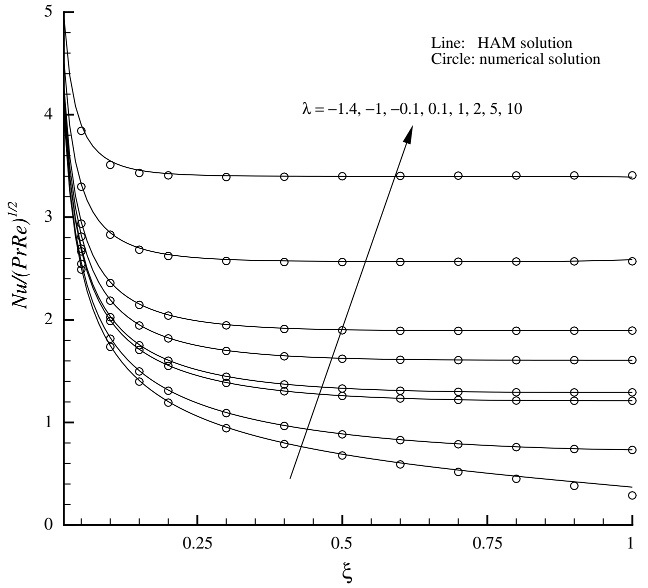

Figure 4. Comparison of the numerical results with the analytic approximation of the local Nusselt number in the whole time period $0 \leq \xi \leq 1$.

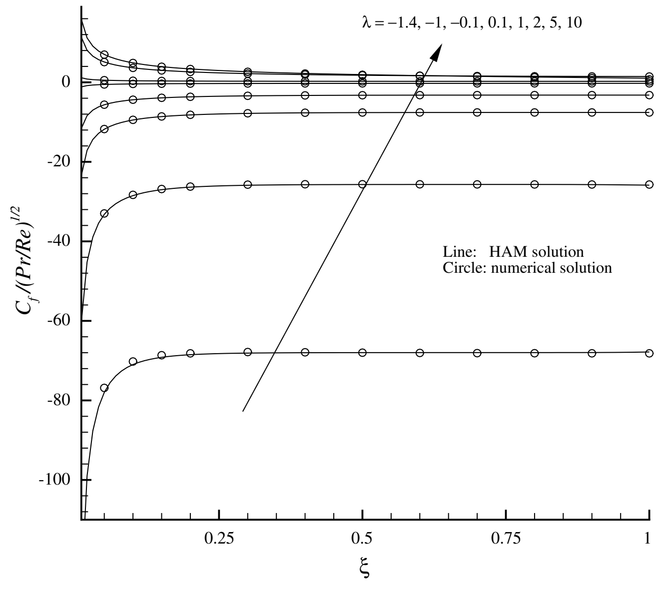

Figure 5. Comparison of the numerical solution with the analytic approximation of the local skin friction coefficient in the whole time period $0 \leq \xi \leq 1$.

Nazar *et al.* (2004), the HAM solutions are uniformly valid for all values of λ as shown in Figure 3. Besides, our analytic solutions agree with numerical results. The variations of the skin friction coefficient C_f and the local Nusselt number Nu in the whole dimensionless time region $0 \le \xi \le 1$ for different values of λ are as shown in Figures 4 and 5, respectively. Note that the 20th-order HAM approximation when $\lambda = -1, -0.1, 0.1, 1, 2, 5,$ and 10, the 25th-order HAM approximation when $\lambda = -1.4$ agree well with the numerical ones. When $\hbar = -1/3$, even the 6th-order HAM result

$$\begin{aligned}
&f''(0,\xi)\\
&\quad = -0.5969568564\lambda - 0.6574056152\lambda\xi - 0.5504690967\lambda^2\xi\\
&\qquad -4.89725117\times10^{-3}\lambda\xi^2 + 0.1255702147\lambda^2\xi^2 + 9.877559992\times10^{-2}\lambda^3\xi^2\\
&\qquad +7.218105469\times10^{-3}\lambda\xi^3 + 2.110725218\times10^{-2}\lambda^2\xi^3 - 1.9023293512\times10^{-2}\lambda^3\xi^3\\
&\qquad -2.262642413\times10^{-2}\lambda^4\xi^3 + 4.052856445\times10^{-3}\lambda\xi^4 + 4.299581885\times10^{-3}\lambda^2\xi^4\\
&\qquad -7.698844961\times10^{-3}\lambda^3\xi^4 + 2.427473049\times10^{-5}\lambda^4\xi^4 + 4.564029788\times10^{-3}\lambda^5\xi^4\\
&\qquad +1.439853516\times10^{-3}\lambda\xi^5 + 3.411321207\times10^{-4}\lambda^2\xi^5 - 2.790427691\times10^{-3}\lambda^3\xi^5\\
&\qquad +9.998905922\times10^{-4}\lambda^4\xi^5 + 8.887303912\times10^{-4}\lambda^5\xi^5 - 7.149633110\times10^{-4}\lambda^6\xi^5\\
&\qquad +1.841088867\times10^{-3}\lambda\xi^6 + 2.047173735\times10^{-4}\lambda^2\xi^6 - 2.487358884\times10^{-3}\lambda^3\xi^6\\
&\qquad +4.243441233\times10^{-4}\lambda^4\xi^6 + 5.060053602\times10^{-4}\lambda^5\xi^6 - 3.408042986\times10^{-4}\lambda^6\xi^6\\
&\qquad +8.4\times10^{-5}\lambda^7\xi^6,
\end{aligned} \tag{52}$$

is valid in the range of $-0.5 \le \lambda \le 5$, as shown in Figure 6. Note that even the 6th-order HAM solutions agree well with the numerical one for *all* time period $0 \le \tau < +\infty$, while the perturbation solution is valid only for small

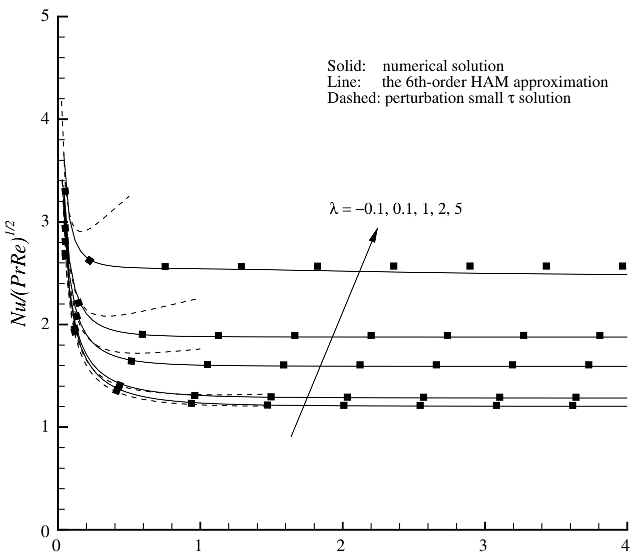

Figure 6. Comparison of the numerical results with the 6th-order approximation of the local Nusselt number. The dashed lines denote the perturbation results (31) for small time variables τ.

time variables and becomes rather bad for large λ such as $\lambda = 1, 2$ and 5, as shown in Figure 6. Thus, by means of the proposed approach, we can obtain accurate analytic solutions uniformly valid for all times $0 \leq \tau < +\infty$ and all possible values of λ.

6. Conclusion

In this paper, we solve the unsteady mixed convection flow near the stagnation point on a heated vertical flat plate embedded in a Darcian fluid-saturated porous medium by means of an analytic technique, namely the HAM. Unlike previous perturbation results, our analytic series solutions are accurate and uniformly valid for all dimensionless times and for all possible values of mixed convection parameter. Besides, all of our analytic solutions agree well with numerical results. Even the lower-order expression (52) of $f''(0, \xi)$ is accurate and valid for all time in some region of λ. To the best of our knowledge, such kind of analytic solutions has never been reported for the considered problems. Furthermore, the study presented also provides us with a new analytic approach to investigate related unsteady problems.

Acknowledgements

Thanks to the "National Science Fund for Distinguished Young Scholars" (Approval No. 50125923) of Natural Science Foundation of China for the financial support. Thanks to the anonymous reviewers for their valuable comments and suggestions.

References

Adomian, G.: 1976, Nonlinear stochastic differential equations, *J. Math. Anal. Appl.* **55**, 441–452.

Aly, E. H., Elliott, L. and Ingham, D. B.: 2003, Mixed convection boundary-layer flow over a vertical surface embedded in a porous medium, *Eur. J. Mech. B/Fluids* **22**, 529–543.

Ayub, M., Rasheed, A. and Hayat, T.: 2003, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, *Int. J. Eng. Sci.* **41**, 2091–2103.

Bejan, A., Dincer, I., Lorente, S.,: Miguel, A. F. and Reis, A. H.: 2004, *Porous and Complex Flow Structures in Modern Technologies*, Springer, New York.

Bejan, A. and Kraus, A. D. (eds): 2003, *Heat Transfer Handbook*, Wiley, New York.

Cebeci, T. and Bradshaw, P.: 1984, *Physcial and Computational Aspects of Convective Heat Transfer*, Springer, New York.

Hayat, T., Khan, M. and Asghar, S.: 2004, Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid, *Acta Mech.* **168**, 213–232.

Harris, S. D., Ingham, D. B. and Pop, I.: 1999, Unsteady mixed convection boundary-layer flow on a vertical surface in a porous medium, *Int. J. Heat Mass Transfer*, **42**, 357–372.

Ingham, D. B., Bejan, A., Mamut, E. and Pop, I. (eds): 2004, *Emerging Technologies and Techniques in Porous Media*, Kluwer, Dordrecht.

Ingham, D. B. and Pop, I. (eds): 1998, *Transport Phenomena in Porous Media,* Pergamon, Oxford.

Johnson, C. H. and Cheng, P.: 1978, Possible similarity solutions for free convection boundary layers adjacent to flat plates in porous media, *Int. J. Heat Mass Transfer* **21**, 709–718.

Karmishin, A. V., Zhukov, A. T. and Kolosov, V. G.: 1990, *Methods of Dynamics Calculation and Testing for Thin-walled Structures*, Mashinostroyenie, Moscow. (in Russian).

Liao, S. J.: 2003a, *Beyond Perturbation: Introduction to Homotopy Analysis Method*, Chapman & Hall/ CRC Press, Boca Raton.

Liao, S. J.: 2003b, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, *J. Fluid Mech.* **488**, 189–212.

Liao, S. J.: 2003c, An explicit analytic solution to the Thomas-Fermi equation, *Appl. Math. Comput.* **144**, 495–506.

Liao, S. J.: 2004, On the homotopy analysis method for nonlinear problems, *Appl. Math. Comput.* **147**, 499–513.

Liao, S. J.: in press, An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate, *Commun. Nonlinear Sci. Numer. Simul.*

Liao, S. J. and Cheung, K. F.: 2003, Homotopy analysis of nonlinear progressive waves in deep water, *J. Eng. Math*. **45**(2), 105–116.

Liao, S. J. and Pop, I.: 2004, Explicit analytic solution for similarity boundary layer equations. *Int. J. Heat Mass Transfer* **47**(1), 75–85.

Lyapunov, A. M.: 1892, *General Problem on Stability of Motion*, Taylor & Francis, London, 1992. (English translation).

Magyari, E., Pop, I. and Keller, B.: 2004, Analytic solutions for unsteady free convection in porous media, *J. Eng. Math.* **48**, 93–104.

Merkin, J. H.: 1980, Mixed convection boundary layer flow on a vertical surface in a saturated porous medium, *J. Eng. Math*, **14**, 301–313.

Nazar, R., Amin, N. and Pop, I.: 2004, Unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface embedded in a porous medium, *Int. J. Heat Mass Transfer* **47**, 2681–2688.

Nield, D. A and Bejan, A.: 1999, *Convection in Porous Media,* 2nd edn, Springer, New York.

Pop, I. and Ingham. D. B.: 2001, *Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Media,* Pergamon, Oxford.

Vafai, K. (ed.): 2000, *Handbook of Porous Media,* Marcel Dekker, New York.

Williams, J. C. and Rhyne, T. H.: 1980, Boundary layer development on a wedge impulsively set into motion, *SIAM J. Appl. Math.* **38**, 215–224.