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ORIGINAL PAPER

A HAM-based analytic approach for physical models with an infinite number of singularities

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Abstract Based on the Homotopy Analysis Method (HAM), an analytic approach is proposed to solve physical models with an infinite number of "singularities". The nonlinear interaction of double cnoidal waves governed by the Korteweg-de Vries (KdV) equation is used to illustrate its validity. The HAM is an analytic technique for highly nonlinear problems, which is based on the homotopy in topology and thus has nothing to do with small physical parameters. Besides, the HAM provides us great freedom to choose proper equation-type and solution-expression for high-order approximation equations. Especially, unlike other methods, the HAM can guarantee the convergence of solution series. Using the HAM, an infinite number of zero denominators of the considered problem are avoided once for all by properly choosing an auxiliary linear operator, as illustrated in this paper. This HAM-based approach has general meanings and can be used to solve many physical problems with lots of "singularities". It also suggests that the so-called "singularity" might not exist physically, but only due to the imperfection of used mathematical methods, because the nature should not contain any singularities at all.

Keywords Singularity · KdV equation · Homotopy analysis method

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1 Introduction

Many physical models have an infinite number of singularities. For example, let us consider the interaction of double cnoidal waves governed by the Korteweg-de Vries (KdV) equation. When one uses perturbation method to solve this problem, an infinite number of zero denominators are encountered. In order to overcome these zero denominators, Haupt [1, 2] used the Stokes' expansion by adding the resonant mode "at the order of resonance with an unknown coefficient to be determined at higher order". However, since there are an infinite number of zero denominators, say, an infinite number of resonant modes, and each of them should be added with an unknown coefficient to be determined, this Stokes' expansion is tiring, and rather complicated.

In this paper, the homotopy analysis method (HAM) [3, 4] is used to proposed an analytic approach to solve physical models with an infinite number of singularities encounted by the perturbation method. The HAM has been successfully applied for nonlinear problems in various areas, such as water waves [5–7], finance [8], nanofluids [9] and so on [10–13]. Different from perturbation method, the HAM is independent of any small/large physical parameter. Unlike other analytic approximation method, it provides us a simple way to guarantee the convergence of the approximation. In particular, the HAM provides us great freedom to choose the auxiliary linear operator [14], which is the key point making it convenient for us to avoid an infinite number of zero denominators in this paper. This is mainly because the HAM transfers the original nonlinear equation into an infinite number of linear sub-problems, governed by the so-called auxiliary linear operators. Thus, as long as the auxiliary linear operators are chosen properly, all zero denominators can be avoided so that these sub-problems can be solved easily.

This paper is organized as follows. The illustrated example is described in §2. The solving procedure of the HAM-based approach is described in§3. Conclusions are presented in §4.

2 An illustrative example

In this paper, we use the Korteweg-de Vries equation in the following form to illustrate how the HAM can overcome an infinite number of zero denominators efficiently:

$$v_t(x,t) + \alpha v_{xxx}(x,t) + \beta v(x,t) v_x(x,t) = 0, \tag{1}$$

which has polycnoidal wave solutions [15], where $\alpha = 1$ and $\beta = 1$ are considered in this paper, $v(x,t)$ describes the wave amplitude affected by both weak nonlinearity and dispersion, and the subscript denotes the partial differentiation with respect to the time t and the space x, respectively. We search for the solution of the double cnoidal wave interaction with time-independent wave amplitude and frequency in the form

$$
\begin{aligned}
v(x,t) &= u(\xi_1,\xi_2) = \sum_{m=1}^{+\infty} a_{m,0}\cos(m\xi_1) + \sum_{n=1}^{+\infty} a_{0,n}\cos(n\xi_2) \\
&+ \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} a_{m,n}\cos(m\xi_1+n\xi_2) + \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} a_{m,-n}\cos(m\xi_1-n\xi_2), \quad (2)
\end{aligned}
$$

which is an approximation of the elliptic function involved in the cnoidal waves, where ξ_1 and ξ_2 are defined by

$$\xi_1 = k_1\, x - \sigma_1\, t + \theta_1, \quad (3)$$

$$\xi_2 = k_2\, x - \sigma_2\, t + \theta_2, \quad (4)$$

$a_{m,n}$ are constants, k_i and σ_i $(i = 1, 2)$ are the wavenumber and angular frequency of the double cnoidal waves with phase difference θ_i, respectively. Then, under the chain rule, the original KdV equation (1) hence reads

$$
\begin{aligned}
&\alpha\left(k_1^3\, u_{\xi_1\xi_1\xi_1} + 3k_1^2k_2u_{\xi_1\xi_1\xi_2} + 3k_1k_2^2u_{\xi_1\xi_2\xi_2} + k_2^3u_{\xi_2\xi_2\xi_2}\right) \\
&\quad -(\sigma_1u_{\xi_1} + \sigma_2u_{\xi_2}) + \beta u\,(k_1u_{\xi_1} + k_2u_{\xi_2}) = 0. \quad (5)
\end{aligned}
$$

Before describing the HAM-based approach for (5), we first introduce the perturbation procedure simply to find out the main reason causing the perturbation method so tiring, i.e. an infinite number of zero denominators.

Let $a_1\cos(\xi_1)$ and $a_2\cos(\xi_2)$ denote the two primary waves. For small wave amplitude, the wave elevation $u(\xi_1,\xi_2)$ and frequency σ_i $(i = 1, 2)$ can be obtained by expanding in terms of a small physical parameter. Without loss of generality, choosing $a_1 = \epsilon$ as the small parameter and $a_2 = \mathcal{O}(\epsilon)$, the solution hence becomes

$$u = \sum_{n=1}^{+\infty}\epsilon^n u_n, \quad \sigma_1 = \sum_{n=0}^{+\infty}\epsilon^n\sigma_{1,n}, \quad \sigma_2 = \sum_{n=0}^{+\infty}\epsilon^n\sigma_{2,n}. \quad (6)$$

Substituting (6) into equation (5) and matching the coefficient of ϵ^n, we have the nth-order perturbation equation:

$$\mathcal{L}[u_n] = F_n(\xi_1,\xi_2) = \sum_i\sum_j d_n^{i,j}\sin(i\xi_1 + j\xi_2), \quad n = 1,2,3,\cdots \quad (7)$$

where $d_n^{i,j}$ are known constants and

$$
\begin{aligned}
\mathcal{L}[f] &= \alpha(k_1^3 f_{\xi_1\xi_1\xi_1} + 3k_1^2k_2 f_{\xi_1\xi_1\xi_2} + 3k_1k_2^2 f_{\xi_1\xi_2\xi_2} + k_2^3 f_{\xi_2\xi_2\xi_2}) \\
&\quad -\sigma_{1,0}f_{\xi_1} - \sigma_{2,0}f_{\xi_2}, \quad (8)
\end{aligned}
$$

with the property

$$\mathcal{L}^{-1}[\sin(i\xi_1 + j\xi_2)] = \frac{1}{\delta_{i,j}}\cos(i\xi_1 + j\xi_2), \quad (9)$$

$$\delta_{i,j} = \alpha(ik_1 + jk_2)^3 + i\sigma_{1,0} + j\sigma_{2,0}. \quad (10)$$

Here $\mathcal{L}^{-1}$ is the inverse operator of $\mathcal{L}$ defined by (8) and $\sigma_{i,0} = -\alpha k_i^3$ $(i = 1, 2)$ are determined by enforcing $d_1^{1,0} = 0$ and $d_1^{0,1} = 0$ in order to balance the 1st-order perturbation equation. Since the linear operator has the property (9), the nth-order perturbation solution becomes

$$u_n = \sum_i \sum_j \frac{d_n^{i,j}}{\delta_{i,j}} \cos(i\xi_1 + j\xi_2), \quad n = 1, 2, 3, \cdots \tag{11}$$

Meanwhile, the nth-order solution of the wave frequencies $\sigma_{1,n}$ and $\sigma_{2,n}$ are determined by enforcing $d_n^{1,0} = 0$ and $d_n^{0,1} = 0$ in the right-hand side of the nth-order perturbation equation (7), since $\delta_{1,0} \equiv 0$ and $\delta_{0,1} \equiv 0$ in (10).

Unfortunately, the perturbation solution (11) breaks down, since there are an infinite number of zero denominators caused by $\delta_{i,j} = 0$, namely that, for any given k_1 and k_2, there are an infinite number of groups (i, j) resulting in $\delta_{i,j} = 0$ for the resonant wavenumber $(ik_1 + jk_2)$. Without loss of generality, let us consider the case

$$k_1 = 1, \quad k_2 = 2. \tag{12}$$

The values of (i, j) corresponding to $\delta_{i,j} = 0$ are shown in Table 1. For lack of space, only the domain of $0 \le i \le 5000$ and $-5000 \le j \le 5000$ are presented although actually there are an infinite number of sets (i, j). These zero denominators result from the corresponding secular terms $\sin(i\xi_1 + j\xi_2)$ in the right-hand side of (7).

3 Analytic approach based on the HAM

Haupt [1, 2] have used the Stokes' expansion to overcome the zero denominators mentioned in §2. In the Stokes' expansion, the resonant mode with wavenumber $(ik_1 + jk_2)$ for $\delta_{i,j} = 0$ should be added "at the order of resonance with an unknown

Table 1 The values of (i, j) corresponding to $\delta_{i,j} = 0$ when $k_1 = 1$ and $k_2 = 2$ in (10)

(i,j)	(i,j)
(0, 1)	(715, −364)
(1, 0)	(896, −455)
(5, −4)	(1105, −560)
(16, −10)	(1344, −680)
(35, −20)	(1615, −816)
(64, −35)	(1920, −969)
(105, −56)	(2261, −1140)
(160, −84)	(2640, −1330)
(231, −120)	(3059, −1540)
(320, −165)	(3520, −1771)
(429, −220)	(4025, −2024)
(560, −286)	(4576, −2300)

coefficient to be determined at higher order", as pointed by Haupt [2]. This procedure must be done for every resonant mode and is rather complex when there exist an infinite number of zero denominators.

In order to avoid an infinite number of zero denominators conveniently, the homotopy analysis method (HAM) is used in this paper. In particular, since HAM provides us great freedom to choose the auxiliary linear operator [14], we can choose such an auxiliary linear operator that the infinite number of zero denominators can be automatically avoided, as shown below.

3.1 Continuous variation

The HAM is based on the homotopy in topology, transforming a nonlinear equation into an infinite number of linear sub-problems by introducing the homotopy-parameter $q \in [0,1]$. We construct such a family of equations, namely the zeroth-order deformation equation:

$$(1-q)\ \bar{\mathcal{L}}\left[U(\xi_1,\xi_2;q)-u_0(\xi_1,\xi_2)\right]=q\ c_0\ \mathcal{N}\left[U(\xi_1,\xi_2;q),\Lambda_1(q),\Lambda_2(q)\right], \quad (13)$$

where $\bar{\mathcal{L}}$ denotes an auxiliary linear operator with the property $\bar{\mathcal{L}}[0]=0$, $c_0 \neq 0$ the convergence-control parameter, $u_0(\xi_1,\xi_2)$ the guess approximation of $u(\xi_1,\xi_2)$, respectively. Here $U(\xi_1,\xi_2;q)$ is a mapping of the unknown function $u(\xi_1,\xi_2)$, and $\Lambda_1(q)$, $\Lambda_2(q)$ are mappings of the unknown angular frequencies σ_1,σ_2, respectively. The nonlinear operator $\mathcal{N}$ defined according to the initial nonlinear equation (5) is

$$\begin{aligned}\mathcal{N}\left[U(\xi_1,\xi_2;q),\Lambda_1(q),\Lambda_2(q)\right]&=\alpha\left(k_1^3\ U_{\xi_1\xi_1\xi_1}+3k_1^2k_2U_{\xi_1\xi_1\xi_2}+3k_1k_2^2U_{\xi_1\xi_2\xi_2}\right.\\ &\left.+k_2^3U_{\xi_2\xi_2\xi_2}\right)-\left(\Lambda_1(q)U_{\xi_1}+\Lambda_2(q)U_{\xi_2}\right)+\beta U\ (k_1U_{\xi_1}+k_2U_{\xi_2}).\end{aligned} \quad (14)$$

As the embedding parameter $q \in [0,1]$ increases from 0 to 1, the mapping function $U(\xi_1,\xi_2;q)$ varies from the guess approximation $u_0(\xi_1,\xi_2)$ to the solution $u(\xi_1,\xi_2)$ of equation (5), so do $\Lambda_1(q)$, $\Lambda_2(q)$ from the guess approximations $\sigma_{1,0},\sigma_{2,0}$ to the angular frequencies σ_1,σ_2, respectively. Note that the guess approximations

$$\sigma_{1,0}=\Lambda_1(0), \quad \sigma_{2,0}=\Lambda_2(0) \quad (15)$$

should be determined. Assume that the Taylor series

$$U(\xi_1,\xi_2;q)\ =u_0(\xi_1,\xi_2)+\sum_{n=1}^{+\infty}u_n(\xi_1,\xi_2)\ q^n, \quad (16)$$

$$\Lambda_1(q)\ =\sigma_{1,0}+\sum_{n=1}^{+\infty}\sigma_{1,n}\ q^n, \quad (17)$$

$$\Lambda_2(q)\ =\sigma_{2,0}+\sum_{n=1}^{+\infty}\sigma_{2,n}\ q^n, \quad (18)$$

exist, where

$$u_n(\xi_1, \xi_2) = \frac{1}{n!} \left. \frac{\partial^n U(\xi_1, \xi_2; q)}{\partial q^n} \right|_{q=0}, \tag{19}$$

$$\sigma_{1,n} = \frac{1}{n!} \left. \frac{\partial^n \Lambda_1(q)}{\partial q^n} \right|_{q=0}, \tag{20}$$

$$\sigma_{2,n} = \frac{1}{n!} \left. \frac{\partial^n \Lambda_2(q)}{\partial q^n} \right|_{q=0}. \tag{21}$$

Note also that we have great freedom to choose the auxiliary linear operator $\bar{\mathcal{L}}$, the guess approximation $u_0(\xi_1, \xi_2)$ and the convergence-control parameter c_0. Assume that all of them are so properly chosen that the Taylor series (16) - (18) are convergent at $q = 1$. Then we have the Mth-order approximation

$$u(\xi_1, \xi_2) \approx u_0(\xi_1, \xi_2) + \sum_{n=1}^{M} u_n(\xi_1, \xi_2), \tag{22}$$

$$\sigma_1 \approx \sigma_{1,0} + \sum_{n=1}^{M} \sigma_{1,n}, \tag{23}$$

$$\sigma_2 \approx \sigma_{2,0} + \sum_{n=1}^{M} \sigma_{2,n}. \tag{24}$$

Till now $u_n(\xi_1, \xi_2), \sigma_{1,n}, \sigma_{2,n}$ are unknown, which will be determined by solving the so-called high-order deformation equation. By differentiating (13) n times with respect to q, then setting $q = 0$ and dividing it by $n!$, we can obtain the nth-order deformation equation:

$$\bar{\mathcal{L}}\left[u_n(\xi_1, \xi_2) - \chi_n\, u_{n-1}(\xi_1, \xi_2)\right] = c_0\, R_n(\xi_1, \xi_2, \vec{\sigma}_{1,n-1}, \vec{\sigma}_{2,n-1}), \tag{25}$$

where

$$\vec{\sigma}_{1,n} = \left\{\sigma_{1,0}, \sigma_{1,1}, \sigma_{1,2}, \cdots, \sigma_{1,n}\right\}, \tag{26}$$

$$\vec{\sigma}_{2,n} = \left\{\sigma_{2,0}, \sigma_{2,1}, \sigma_{2,2}, \cdots, \sigma_{2,n}\right\}. \tag{27}$$

$$\begin{aligned}
&R_n(\xi_1, \xi_2, \vec{\sigma}_{1,n-1}, \vec{\sigma}_{2,n-1}) \\
&= \frac{1}{(n-1)!} \left. \left\{ \frac{d^{n-1}\mathcal{N}\left[U(\xi_1, \xi_2; q), \Lambda_1(q), \Lambda_2(q)\right]}{dq^{n-1}} \right\} \right|_{q=0} \\
&= \alpha \left(k_1^3 \frac{\partial^3 u_{n-1}}{\partial \xi_1 \partial \xi_1 \partial \xi_1} + 3k_1^2 k_2 \frac{\partial^3 u_{n-1}}{\partial \xi_1 \partial \xi_1 \partial \xi_2} + 3k_1 k_2^2 \frac{\partial^3 u_{n-1}}{\partial \xi_1 \partial \xi_2 \partial \xi_2} + k_2^3 \frac{\partial^3 u_{n-1}}{\partial \xi_2 \partial \xi_2 \partial \xi_2} \right) \\
&- \sum_{j=0}^{n-1} \left(\sigma_{1,j} \frac{\partial u_{n-1-j}}{\partial \xi_1} + \sigma_{2,j} \frac{\partial u_{n-1-j}}{\partial \xi_2} \right) \\
&+ \beta \sum_{j=0}^{n-1} u_j \left(k_1 \frac{\partial u_{n-1-j}}{\partial \xi_1} + k_2 \frac{\partial u_{n-1-j}}{\partial \xi_2} \right)
\end{aligned} \tag{28}$$

with the definition

$$\chi_n = \begin{cases} 0, & \text{when } n \le 1, \\ 1, & \text{when } n > 1. \end{cases} \tag{29}$$

It should be emphasized that the nth-order deformation equation (25) is *linear* with the *known* right-hand side term and thus is easy to solve successively.

3.2 Auxiliary linear operator

Before solving the high-order deformation equation (25) to obtain $u_n(\xi_1, \xi_2)$, $\sigma_{1,n}$ and $\sigma_{2,n}$, most important of all, the auxiliary linear operator $\bar{\mathcal{L}}$ should be determined first. As mentioned in §3, equation (5) can not be solved by the perturbation method since there are inevitably an infinite number of zero denominators, as shown in Table 1. Thus, the linear operator can not simply be the linear part of equation (5). It should be mentioned that although Haupt [1, 2] gave some treatments to overcome these zero denominators, the procedure is rather complex especially when the number of the zero denominators is infinite. However, the HAM can avoid these an infinite number of zero denominators easily and effectively by choosing a proper auxiliary linear operator, as illustrated in this section.

Since HAM provides us great freedom to choose the auxiliary linear operator, in the general case of double cnoidal waves, we can choose the following auxiliary linear operator

$$\bar{\mathcal{L}}u = \alpha \left[k_1^3 \left(u_{\xi_1\xi_1\xi_1} + u_{\xi_1} \right) + c_1\, k_2^3 \left(u_{\xi_2\xi_2\xi_2} + u_{\xi_2} \right) \right], \quad k_1 > 0, \quad k_2 > 0, \tag{30}$$

where $c_1 \neq 0$ is the second convergence-control parameter (the first one is c_0). This linear operator has the properties:

$$\bar{\mathcal{L}}[\cos(m\xi_1 + n\xi_2)] = -\bar{\delta}_{i,j} \sin(m\xi_1 + n\xi_2). \tag{31}$$

where

$$\bar{\delta}_{i,j} = \alpha \left[k_1^3\, m(1 - m^2) + c_1\, k_2^3\, n(1 - n^2) \right], \quad \alpha \neq 0. \tag{32}$$

Note that, in the case of $m = 0$ or $m = \pm 1$, and $n = 0$ or $n = \pm 1$, we have

$$\bar{\mathcal{L}}\,[\cos(\xi_1)] = \bar{\mathcal{L}}\,[\cos(\xi_2)] = \bar{\mathcal{L}}\,[\cos(\xi_1 + \xi_2)] = \bar{\mathcal{L}}\,[\cos(\xi_1 - \xi_2)] = 0. \tag{33}$$

And the inverse operator $\bar{\mathcal{L}}^{-1}$ has the following property:

$$\bar{\mathcal{L}}^{-1}[\sin(m\xi_1 + n\xi_2)] = -\frac{1}{\bar{\delta}_{i,j}} \cos(m\xi_1 + n\xi_2), \tag{34}$$

where $m(1 - m^2) = 0$ and $n(1 - n^2) = 0$ in (32) do not hold at the same time. It should be emphasized that we have great freedom to choose the value of the second convergence-control parameter c_1 so that it is easy for us to ensure

$$\bar{\delta}_{i,j} \neq 0$$

for *all* integers m, n while

$$m(1 - m^2) \neq 0, \;\; n(1 - n^2) \neq 0.$$

For example, when $m(1-m^2)\neq 0$ and $n(1-n^2)\neq 0$, we choose c_1 as an irrational number, such as π, $\sqrt{2}$ and so on. Then, $k_1^3 m(1-m^2)\neq 0$ is a non-zero rational number, but $c_1\, k_2^3\, n(1-n^2)$ is a non-zero irrational one, therefore it holds

$$k_1^3\, m(1-m^2)+c_1\, k_2^3\, n(1-n^2)\neq 0$$

for all possible integers m and n except $m=0, n=1$, or $m=1, n=0$, or $m=1, n=\pm 1$, because the sum of a non-zero rational number and a non-zero irrational number is always a non-zero irrational number. It should be emphasized that different values of c_1 give the same result. For this reason and without loss of generality, we choose $c_1=\pi/3$. In this way, the an infinite number of zero denominators mentioned in §2 can be avoided automatically and no more treatments need to be done.

3.3 Solution of the high-order deformation equation

In this part, after the auxiliary linear operator is determined, the high-order deformation equation (25) is solved in order to get the solution of $u_n(\xi_1,\xi_2)$, $\sigma_{1,n}$ and $\sigma_{2,n}$. Recalling that we are searching for the solutions expressed by (2) and

$$\bar{\mathcal{L}}\left[B_1\cos(\xi_1)+B_2\cos(\xi_2)+B_3\cos(\xi_1+\xi_2)+B_4\cos(\xi_1-\xi_2)\right]=0$$

for arbitrary constants B_1, B_2, B_3 and B_4, we choose the following guess approximation

$$u_0(\xi_1,\xi_2)=a_1\cos(\xi_1)+a_2\cos(\xi_2)+\lambda_0\, a_1a_2\cos(\xi_1+\xi_2)+\mu_0\, a_1a_2\cos(\xi_1-\xi_2), \tag{35}$$

where a_1 and a_2 are given, but λ_0 and μ_0 are unknown constants related to the wave interactions whose values will be determined later. Note that it holds

$$\bar{\mathcal{L}}u_0(\xi_1,\xi_2)=0.$$

More importantly, as shown below, the unknown constants λ_0 and μ_0 must be introduced here so as to avoid the secular terms in $u(\xi_1,\xi_2)$.

When $n=1$, it is found that

$$R_1(\xi_1,\xi_2,\vec{\sigma}_{1,0},\vec{\sigma}_{2,0})=\sum_{i=0}^{2}\sum_{j=-2}^{2}B_1^{i,j}\ \sin(i\ \xi_1+j\ \xi_2), \tag{36}$$

where

$$B_1^{1,0}\ =a_1\left[\alpha k_1^3-\frac{1}{2}\beta k_1(\lambda_0+\mu_0)a_2^2+\sigma_{1,0}\right], \tag{37}$$

$$B_1^{0,1}\ =a_2\left[\alpha k_2^3-\frac{1}{2}\beta k_2(\lambda_0+\mu_0)a_1^2+\sigma_{2,0}\right], \tag{38}$$

$$B_1^{1,1}\ =-a_1a_2\left[\frac{1}{2}\beta(k_1+k_2)-\alpha\lambda_0(k_1+k_2)^3-\lambda_0(\sigma_{1,0}+\sigma_{2,0})\right], \tag{39}$$

$$B_1^{1,-1}\ =-a_1a_2\left[\frac{1}{2}\beta(k_1-k_2)-\alpha\mu_0(k_1-k_2)^3-\mu_0(\sigma_{1,0}-\sigma_{2,0})\right], \tag{40}$$

$$\vdots$$

According to (34), to avoid the so-called secular terms, the coefficients $B_1^{1,0}$, $B_1^{0,1}$, $B_1^{1,1}$ and $B_1^{1,-1}$ must be zero, which give the set of algebraic equations

$$\alpha k_1^3 - \frac{1}{2}\beta k_1(\lambda_0 + \mu_0)a_2^2 + \sigma_{1,0} = 0, \tag{41}$$

$$\alpha k_2^3 - \frac{1}{2}\beta k_2(\lambda_0 + \mu_0)a_1^2 + \sigma_{2,0} = 0, \tag{42}$$

$$\frac{1}{2}\beta(k_1 + k_2) - \alpha\lambda_0(k_1 + k_2)^3 - \lambda_0(\sigma_{1,0} + \sigma_{2,0}) = 0, \tag{43}$$

$$\frac{1}{2}\beta(k_1 - k_2) - \alpha\mu_0(k_1 - k_2)^3 - \mu_0(\sigma_{1,0} - \sigma_{2,0}) = 0. \tag{44}$$

According to the above algebraic equations, we can have the solution of $\sigma_{1,0}$, $\sigma_{2,0}$, λ_0 and μ_0. Then, the corresponding particular solution is given by

$$\begin{aligned} u_1^*(\xi_1, \xi_2) \;=\; & \bar{\mathcal{L}}^{-1}\left[c_0\ R_1(\xi_1, \xi_2, \sigma_{1,0}, \sigma_{2,0})\right] \\ =\; & \frac{c_0\ B_1^{2,0}}{6\alpha k_1^3}\cos(2\ \xi_1) + \frac{c_0\ B_1^{0,2}}{6\alpha c_1 k_2^3}\cos(2\ \xi_2) + \frac{c_0\ B_1^{1,2}}{6\alpha c_1 k_2^3}\cos(\xi_1 + 2\xi_2) \\ & + \frac{c_0\ B_1^{2,1}}{6\alpha k_1^3}\cos(2\xi_1 + \xi_2) + \frac{c_0\ B_1^{2,2}}{6\alpha(k_1^3 + c_1 k_2^3)}\cos(2\xi_1 + 2\xi_2) \\ & - \frac{c_0\ B_1^{1,-2}}{6\alpha c_1 k_2^3}\cos(\xi_1 - 2\xi_2) + \frac{c_0\ B_1^{2,-1}}{6\alpha k_1^3}\cos(2\xi_1 - \xi_2) \\ & + \frac{c_0\ B_1^{2,-2}}{6\alpha(k_1^3 - c_1 k_2^3)}\cos(2\xi_1 - 2\xi_2). \end{aligned} \tag{45}$$

And the general solution of $u_1(\xi_1, \xi_2)$ is given by

$$\begin{aligned} u_1(\xi_1, \xi_2) \;=\; & u_1^*(\xi_1, \xi_2) + A_{1,1}\cos(\xi_1) + A_{1,2}\cos(\xi_2) \\ & + \lambda_1 a_1 a_2 \cos(\xi_1 + \xi_2) + \mu_1 a_1 a_2 \cos(\xi_1 - \xi_2). \end{aligned} \tag{46}$$

Because the primary wave components $a_1\cos(\xi_1)$ and $a_2\cos(\xi_2)$ are known, thus the coefficients $A_{1,1}$ and $A_{1,2}$ must be zero. Therefore, we have

$$u_1(\xi_1, \xi_2) \;=\; u_1^*(\xi_1, \xi_2) + \lambda_1 a_1 a_2 \cos(\xi_1 + \xi_2) + \mu_1 a_1 a_2 \cos(\xi_1 - \xi_2), \tag{47}$$

where λ_1 and μ_1 are unknown, which can be determined later in a similar way.

In the general case of $n \geq 1$, it is found that

$$R_n = \sum_{i=0}^{n+1}\sum_{j=-n-1}^{n+1} B_n^{i,j}\ \sin(i\ \xi_1 + j\ \xi_2), \tag{48}$$

and the unknown coefficients $\sigma_{1,n-1}, \sigma_{2,n-1}, \lambda_{n-1}, \mu_{n-1}$ are determined by a set of algebraic equations

$$B_n^{1,0} = 0, \quad B_n^{0,1} = 0, \quad B_n^{1,1} = 0, \quad B_n^{1,-1} = 0. \tag{49}$$

Similarly, we can solve the linear high-order deformation equation (25) successively, in the order $n = 1, 2, 3, \cdots$. By means of the inverse operator (34) and using the

Table 2 The solution of equations (41)-(44) when the guess approximation is chosen as (35) for the case of $k_1 = 1, k_2 = 2, a_1 = 1/2, a_2 = 1/10$ and $c_1 = \pi/3$

i	λ_0	μ_0	$\sigma_{1,0}$	$\sigma_{2,0}$
group 1	0.0833333	−0.0833333	−1.00000	−8.00000
group 2	0.0618146	24.5112	−0.877135	−1.85674
group 3	−70.65	−0.0214459	−1.35336	−25.6679

symbolic computation software such as Mathematica, it is easy to get solutions of such kind of linear equations.

It should be mentioned that there are only four homogeneous solutions of the auxiliary linear operator (30). In other words, in the high-order deformation equation (25) there are only four correspondingly secular terms which can be avoided by enforcing their coefficients to be zero. This just can determine the unknown quantities of λ_n, μ_n, $\sigma_{1,n}$ and $\sigma_{2,n}$. In this process, the infinite number of zero denominators mentioned in §2 disappear automatically.

3.4 Results analysis

In this section, the convergent-control parameter c_0 is determined through the residuals of equation (5) to ensure the convergence of the HAM approximations (22)-(24). The convergent wave angular frequencies σ_1 and σ_2 are presented in detail. In addition, the energy distribution of the wave system is investigate as well.

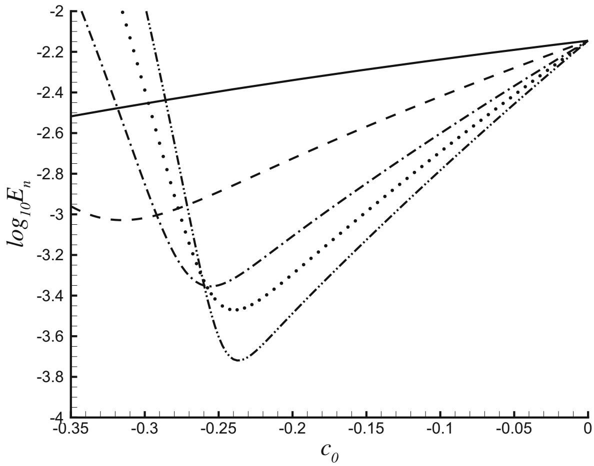

Fig. 1 The averaged residual squares versus c_0 in the case of $a_1 = 1/2$, $a_2 = 1/10$, $k_1 = 1$, $k_2 = 2$ by means of $c_1 = \pi/3$. Solid line: 1st-order approximation; Dashed line: 3rd-order approximation; Dash-dotted line: 5th-order approximation; Dot: 6th-order approximation; Dash-dot-dotted line: 7th-order approximation

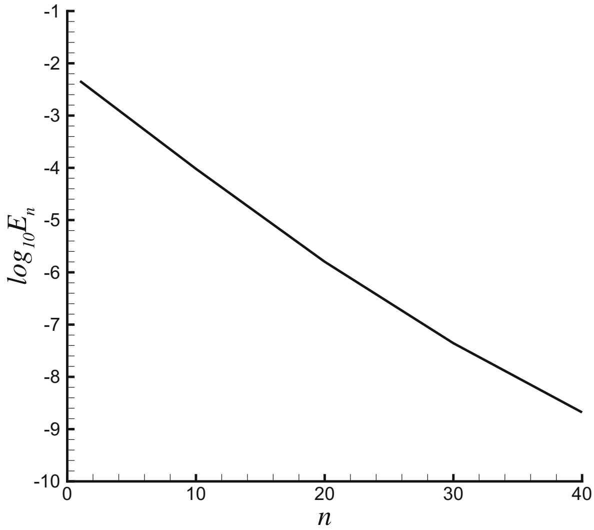

Fig. 2 The averaged residual squares versus the approximation order n in the case of $a_1 = 1/2$, $a_2 = 1/10$, $k_1 = 1$, $k_2 = 2$ by means of $c_0 = -0.2$ and $c_1 = \pi/3$

3.4.1 Validation of the HAM solution

Let us first consider the case

$$a_1 = 1/2, \quad a_2 = 1/10, \quad k_1 = 1, \quad k_2 = 2. \tag{50}$$

The solution of the algebraic equations (41)-(44) is shown in Table 2. Take group 1 in Table 2 as an example to illustrate the determination of the unknown convergent-control parameter c_0. Define the averaged residual square as

$$E_n = \frac{1}{(1+K)^2}\sum_{i=0}^{K}\sum_{j=0}^{K}\left\{\mathcal{N}\left[u(i\Delta\xi_1, j\Delta\xi_2), \sum_{p=0}^{n}\sigma_{1,p}, \sum_{p=0}^{n}\sigma_{2,p}\right]\right\}^2$$

at the nth-order of approximation, where

$$\Delta\xi_1 = \Delta\xi_2 = \frac{\pi}{M},$$

Table 3 The averaged residual squares in the case of $k_1 = 1$, $k_2 = 2$, $a_1 = 1/2$, $a_2 = 1/10$ by means of $c_0 = -0.2$ and $c_1 = \pi/3$

n(order of approximation)	E_n
1	4.57×10^{-3}
10	9.00×10^{-5}
20	1.31×10^{-6}
30	2.48×10^{-8}
40	2.07×10^{-9}

Table 4 The angular frequencies in the case of $k_1 = 1, k_2 = 2, a_1 = 1/2, a_2 = 1/10$ by means of $c_0 = -0.2$ and $c_1 = \pi/3$

n	σ_1	σ_2
10	−0.99070	−7.99986
20	−0.98970	−7.99981
30	−0.98959	−7.99979
40	−0.98958	−7.99978
45	−0.98958	−7.99978
50	−0.98958	−7.99978

and $\mathcal{N}$ is defined by (14). Here, we use $M = 10$ in this paper. The residual square E_n is dependent upon c_0, as shown in Fig. 1. As n (the order of approximation) increases, the residual square E_n decreases in the region of $-0.26 < c_0 < 0$, which defines a domain $\mathbf{c}_c$ corresponding to convergent series solutions, namely that the series (22) - (24) are convergent as long as we choose any value of $c_0 \in \mathbf{c}_c$. In this way, we can ensure the convergence of approximation. This approach is called "optimal homotopy analysis" [16–18]. A general optimal approach with infinite convergent-control parameters can be found in Niu [19] and Liao [4]. The averaged residual square E_n decreases as the order of approximation increases by means of $c_0 = -0.2$, as shown by Fig. 2 and Table 3. The corresponding angular frequencies σ_1 and σ_2 converge as shown in Table 4. It should be emphasized that the HAM result agrees well with the 3rd-order perturbation solution given by the Stokes expansion [2], as presented in Table 5 and 6 for different amplitudes, where $\sigma_1^{Stokes} = k_1(-1 + a_1^2/24)$, $\sigma_2^{Stokes} = k_2(-4 + a_2^2/96)$. The solutions at different time when $k_1 = 1, k_2 = 2$, $a_1 = 1/2, a_2 = 1/4$ are plotted in Fig. 3.

Meanwhile, it should be mentioned that, when it is computed to higher orders, there is no convergent solutions for group 2 and 3 in Table 2.

Table 5 The angular frequencies of the HAM and Stokes expansion versus a_2 in the case of $k_1 = 1, k_2 = 2, a_1 = 1/2$

a_2	σ_1^{HAM}	σ_1^{Stokes}	σ_2^{HAM}	σ_2^{Stokes}
0.1	−0.98958	−0.98958	−7.99978	−7.99979
0.2	−0.98958	−0.98958	−7.99915	−7.99917
0.4	−0.98958	−0.98958	−7.99663	−7.99667
0.6	−0.98957	−0.98958	−7.99244	−7.99250
0.8	−0.98956	−0.98958	−7.98657	−7.98667
1.0	−0.98956	−0.98958	−7.98657	−7.97917
1.2	−0.98954	−0.98958	−7.96980	−7.97000
1.4	−0.98943	−0.98958	−7.95889	−7.95917
1.6	−0.98933	−0.98958	−7.94631	−7.94667

Table 6 The angular frequencies of the HAM and Stokes expansion versus a_1 in the case of $k_1 = 1, k_2 = 2, a_2 = 1/2$

a_1	σ_1^{HAM}	σ_1^{Stokes}	σ_2^{HAM}	σ_2^{Stokes}
0.1	−0.99958	−0.99958	−7.99479	−7.99479
0.2	−0.99833	−0.99833	−7.99479	−7.99479
0.4	−0.99333	−0.99333	−7.99476	−7.99479
0.6	−0.98499	−0.98500	−7.99471	−7.99479
0.8	−0.97329	−0.97333	−7.99460	−7.99479
1.0	−0.95824	−0.95833	−7.99441	−7.99479
1.2	−0.93983	−0.94000	−7.99408	−7.99479
1.4	−0.91803	−0.91833	−7.99360	−7.99479
1.6	−0.89282	−0.89333	−7.99281	−7.99479

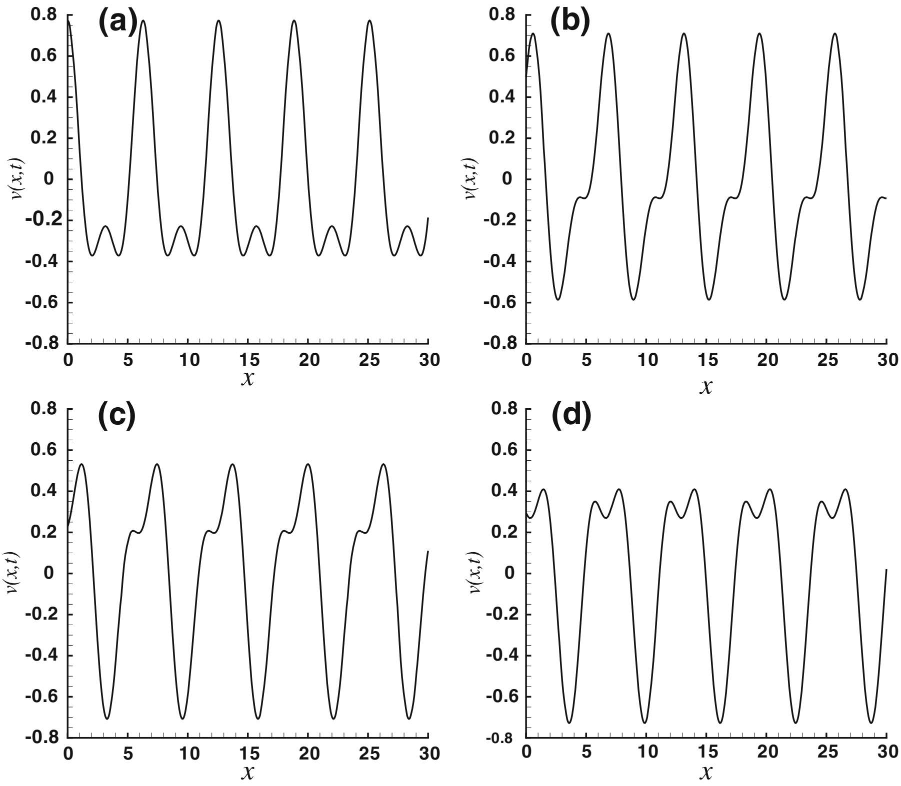

Fig. 3 The solution of $v(x, t)$ in the case of $a_1 = 1/2, a_2 = 1/4, k_1 = 1, k_2 = 2$. (**a**): $t = 0$; (**b**): $t = 0.2$; (**c**): $t = 0.4$; (**d**): $t = 0.5$

Table 7 The energy distribution of the wave system in the case of $a_1 = 1/2, a_2 = 1/10, k_1 = 1, k_2 = 2$ by means of $c_0 = -0.2$ and $c_1 = \pi/3$

(m, n)	$\mathcal{E}_{m,n}$	Annotation
(1,0)	0.9598	primary wave
(0,1)	3.84×10^{-2}	primary wave
(2,0)	1.66×10^{-3}	
(1,−1)	6.66×10^{-5}	
(1,1)	6.67×10^{-5}	
(0,2)	1.65×10^{-7}	
(3,0)	1.62×10^{-6}	
(2,1)	9.12×10^{-8}	
(1,−2)	5.86×10^{-9}	
(1,2)	5.53×10^{-10}	
(0,3)	3.92×10^{-13}	
...	...	
(5,−4)	2.45×10^{-34}	resonant wave
...	...	
(16,−10)	3.17×10^{-112}	resonant wave
...	...	

3.4.2 *Energy distribution of the wave system*

We define

$$\Pi = \sum_{n=1}^{+\infty} a_{0,n}^2 + \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} a_{m,n}^2, \quad \mathcal{E}_{m,n} = a_{m,n}^2/\Pi, \tag{51}$$

where Π denotes the total energy of the wave system and $\mathcal{E}_{m,n}$ the energy of the mode $\cos(m\xi_1 + n\xi_2)$. The energy distribution of different wave modes is presented in Table 7. It is remarkable that the resonant components (corresponding to the wavenumber $(ik_1 + jk_2)$ for $\delta_{i,j} = 0$) have nothing special compared with other harmonics in the energy distribution, namely that the higher-order harmonics have less wave energy than the lower-order ones. For sake of space, only the first two resonant wave components are shown in Table 7.

4 Concluding remarks

In the frame of the HAM, the infinite number of zero denominators encountered by perturbation method can be avoided conveniently by choosing a proper auxiliary linear operator for high-order approximation equations, as illustrated by the the KdV equation. Mathematically, the HAM-based approach is much easier and convenient than the Stokes expansion, since we do not need to add terms with to-be-determined coefficients for each singularity [1, 2]. This HAM-based approach has general meanings and can be used to solve many physical problems with an infinite number of singularities.

It should be mentioned that such kind of "singularities" are induced by perturbation method, but do not really exist physically. In this paper, we take the KdV equation as an example to illustrate that the HAM-based approach is effective and convenient, even if there exist such kind of an infinite number of "singularities". This is mainly because the HAM provides us great freedom to choose a much better auxiliary linear operator than the original ones. It also illustrates the great potential and advantages of the HAM with comparison to perturbation techniques. Possibly, using this kind of freedom of the HAM, lots of complicated, difficult nonlinear problems might be solved in more convenient ways.

In addition, it also suggests that, in general cases, the so-called "singularities" might not exist in nature at all, but only due to the imperfection of used mathematical methods, because the nature should not contain any singularities!

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