Z. angew. Math. Phys. 57 (2006) 777–792
0044-2275/06/050777-16
DOI 10.1007/s00033-006-0061-x

Zeitschrift für angewandte Mathematik und Physik ZAMP

Exponentially decaying boundary layers as limiting cases of families of algebraically decaying ones

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Abstract. The boundary value problem for the similar stream function $f = f(\eta; \lambda)$ of the Cheng–Minkowycz free convection flow over a vertical plate with a power law temperature distribution $T_w(x) = T_\infty + Ax^\lambda$ in a porous medium is revisited. It is shown that in the λ-range $-1/2 < \lambda < 0$, the well known exponentially decaying "first branch" solutions for the velocity and temperature fields are not some isolated solutions as one has believed until now, but limiting cases of families of algebraically decaying multiple solutions. For these multiple solutions well converging analytical series expressions are given. This result yields a bridging to the historical quarreling concerning the feasibility of exponentially and algebraically decaying boundary layers. Owing to a mathematical analogy, our results also hold for the similar boundary layer flows induced by continuous surfaces stretched in viscous fluids with power-law velocities $u_w(x) \sim x^\lambda$.

Mathematics Subject Classification (2000). 76D10, 76S05, 76M99.

Keywords. Boundary layers, similarity, porous media, algebraic decay, homotopy analysis.

1. Introduction and summary of the previous results

We consider the boundary value problem

$$
\begin{gathered}
f''' + \frac{\lambda+1}{2} f f'' - \lambda f'^2 = 0 \\
f(0;\lambda) = 0, f'(0,\lambda) = 1, f'(\infty;\lambda) = 0
\end{gathered}
\tag{1}
$$

for the similar stream function $f = f(\eta; \lambda)$ of the free convection boundary layer flow over a vertical plate immersed in a porous medium as being first formulated by Cheng and Minkowycz, [1]. The plate is assumed impermeable and its temperature distribution is of the power-law form, $T_w(x) = T_\infty + Ax^\lambda$. The velocity components and the temperature field are given in terms of $f(\eta)$ as follows

$$
\begin{aligned}
u &= [\rho_\infty g \beta K (T_w - T_\infty)/\mu] f'(\eta;\lambda) \\
v &= \frac{1}{2}[\alpha \rho_\infty g \beta K (T_w - T_\infty)/\mu x]^{1/2} [(1-\lambda)\eta f' - (1+\lambda) f] \\
T &= T_\infty + (T_w - T_\infty) f'(\eta;\lambda)
\end{aligned}
\tag{2}
$$

(everywhere the original notation of Cheng and Minkowycz is used).

Cheng and Minkowycz, [1], solved the problem (1) numerically in the range $1/3 < \lambda < 1$. A comprehensive study has then performed by Ingham and Brown, [2]. Later, several analytical and numerical solutions of this problem were collected and discussed in the monograph of Pop and Ingham [3]. The main results reported by Ingham and Brown, [2] may be summarized as follows.

1. The Cheng–Minkowycz problem (1) does admit solutions only in the parameter range $\lambda > -1/2$.
2. In the range $-1/2 < \lambda < 0$ both f and f' are non-negative for all $\eta \geq 0$.
3. For $\lambda = +1$ and $\lambda = -1/3$ elementary analytical solutions exist (see also below).
4. For $\lambda > 1$ a second branch of solutions exists.
5. For the dimensionless wall temperature gradient $f''(0;\lambda)$ the integral relationship

$$f''(0;\lambda) = -\frac{3\lambda+1}{2}\int_0^\infty f'^2(\eta;\lambda)d\eta \tag{3}$$

   holds. However, it is important to underline here that this relationship is valid only for the solutions which satisfy the asymptotic condition

$$\lim_{\eta\to\infty} f(\eta;\lambda)f'(\eta;\lambda) = 0. \tag{4}$$

   Obviously, the exponentially decaying solutions $f \equiv f_{\exp}(\eta;\lambda)$ satisfy this condition. Thus, for these solutions Eq. (3) implies

$$f''_{\exp}(0;\lambda) = 0 \quad \text{for} \quad \lambda = -\frac{1}{3} \tag{5a}$$

   and

$$\mathrm{sgn}\left[f''_{\exp}(0;\lambda)\right] = -\mathrm{sgn}\left(\lambda + \frac{1}{3}\right) \quad \text{for } \lambda \neq -\frac{1}{3}. \tag{5b}$$

   The dependence of $f''_{\exp}(0,\lambda)$ on λ, as calculated by Ingham and Brown [2] is shown in Fig. 1. The domain of existence of the algebraically decaying solutions, the main issue of the present paper, is also shown in Fig. 1. The details will be discussed in Sections 2 and 3 below.
6. Ingham and Brown [2] also gave valuable estimates of $f''_{\exp}(0;\lambda)$ for $0 < \lambda + 0.5 << 1$ as well as for λ near to zero. These are:

$$f''_{\exp}(0;\lambda) = 0.078103 \cdot (\lambda + 0.5)^{-3/4} \tag{6}$$

   for $0 < \lambda + 0.5 << 1$, and

$$f''_{\exp}(0;\lambda) = -0.44375 - 0.85665 \cdot \lambda + 0.66943\lambda^2 \tag{7}$$

   for $|\lambda| << 1$, respectively.

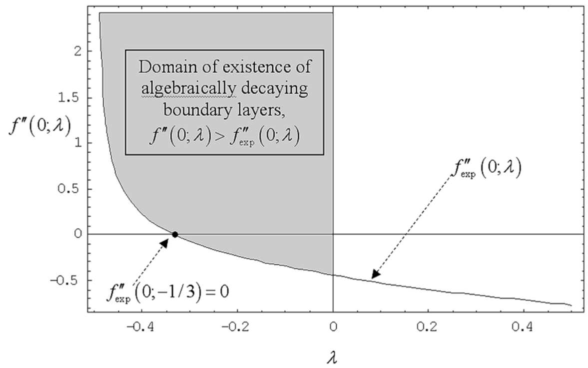

Figure 1. In the range $-1/2 < \lambda < 0$, the whole domain $f''(0;\lambda) > f''_{\exp}(0;\lambda)$ above the Ingham–Brown characteristic curve $f''_{\exp}(0;\lambda)$ corresponding to the exponentially decaying solutions is densely "filled" with values of $f''(0;\lambda)$ corresponding to algebraically decaying solutions.

A comprehensive analytical and numerical investigation (for a slightly rescaled form) of the boundary value problem (1) as it occurs in the context of the boundary layer flows induced by continuous surfaces stretched with power-law velocities has been reported by Banks, [4].

Recently for the exponentially decaying solutions $f \equiv f_{\exp}(\eta;\lambda)$ of the problem (1) analytical expressions in form of infinite series with controllable convergence have been given by Liao and Pop, [5] by applying the homotopy analysis method (Liao, [6]). This method allows for the calculation of the similar wall temperature gradient $f''_{\exp}(0;\lambda)$ and of the similar entrainment velocity $f_{\exp}(\infty;\lambda)$ to any desired precision.

2. Algebraically decaying solutions

2.1. General features

We first examine the general question of the existence in the parameter range $-1/2 < \lambda < 0$ of similar velocity and temperature profiles $f'(\eta;\lambda)$ with algebraic asymptotic decay of the form

$$f'(\eta;\lambda) \sim \eta^b \quad \text{as} \quad \eta \to \infty \tag{8a}$$

which yields

$$f(\eta;\lambda) \sim \frac{1}{b+1}\eta^{b+1} \quad \text{as} \quad \eta \to \infty \tag{8b}$$

where b is a constant. As a requirement of the boundary condition $f'(\infty;\lambda) = 0$, the exponent b must be negative. Substituting (8) in Eq. (1) and balancing the

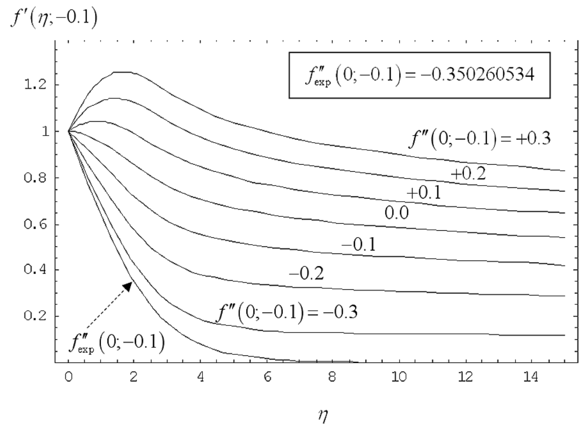

Figure 2. Here the exponentially decaying and seven (presumably) algebraically decaying dimensionless temperature (and velocity) profiles $f'_{\exp}(\eta;\lambda)$ and $f'(\eta;\lambda)$ are shown for $\lambda = -0.1$. The exponentially decaying profile corresponds to the value $f''_{\exp}(0;-0.1) = -0.350260534$ of the dimensionless wall temperature gradient. As $f''(0;-0.1)$ approaches the value of $f''_{\exp}(0;-0.1)$, the family of algebraically decaying profiles goes over continuously in the exponentially decaying one.

dominant terms, we obtain that in the range $-1/2 < \lambda < 0$ which we are interested in, the asymptotic behavior (8) is possible for $b = 2\lambda/(1-\lambda)$ which further implies

$$b + 1 \equiv \beta = \frac{1+\lambda}{1-\lambda} \quad \text{with } 0 < \beta < 1. \tag{9}$$

Substituting Eqs. (8) in Eq. (4) we easily deduce that this condition is satisfied by the algebraically decaying solutions only in the range $-1/2 < \lambda < -1/3$ and it is violated in the remaining part $-1/3 \leq \lambda < 0$ of the interval of interest $-1/2 < \lambda < 0$. Hence, Eq. (5b) which is a consequence of Eq. (3), holds also for our algebraically decaying solutions, but only in the range $-1/2 < \lambda < -1/3$ where $f''(0;\lambda) > f''_{\exp}(0;\lambda) > 0$. For $-1/3 \leq \lambda < 0$ where $f''_{\exp}(0;\lambda) \leq 0$, in the existence domain $f''(0;\lambda) > f''_{\exp}(0;\lambda)$ of the algebraically decaying solutions both negative and positive values of $f''(0;\lambda)$ are possible.

The numerical "proof" for the existence domain $f''(0;\lambda) > f''_{\exp}(0;\lambda)$ of the multiple solutions for $-1/2 < \lambda < 0$ is straightforward. It is illustrated in Fig. 2 where the exponentially decaying and a couple of (presumably) algebraically decaying dimensionless temperature (and velocity) profiles $f'_{\exp}(\eta;\lambda)$ and $f'(\eta;\lambda)$, respectively, are shown for $\lambda = -0.1$. All these profiles have been obtained by a direct numerical solution of the problem (1). The exponentially decaying solution corresponds to the value $f''_{\exp}(0;-0.1) = -0.350260534$ of the dimensionless wall temperature gradient. All the values $f''(0;-0.1) > f''_{\exp}(0;-0.1) = -0.350260534$ furnish (presumably) algebraically decaying solutions of the problem (1).

We underline again that the plots of Fig. 2 only show that the asymptotic decay of the solutions corresponding to values $f''(0;-0.1) > f''_{\exp}(0;-0.1)$ is slower

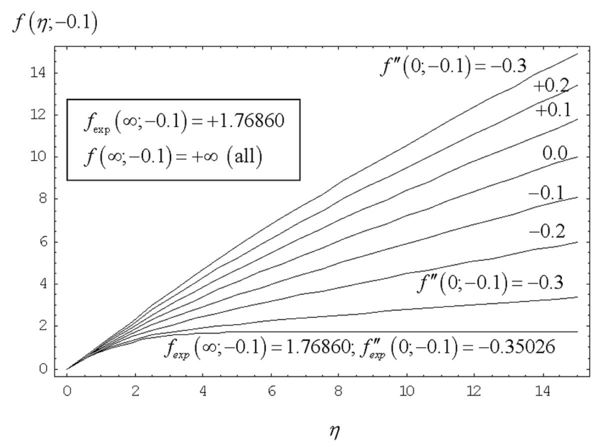

Figure 3. Plots of the stream functions $f(\eta;\lambda)$ corresponding to the velocity profiles of Fig. 2. All the curves corresponding to $f''(0;-0.1) > f''_{\rm exp}(0;-0.1)$ go to infinity as $\eta \to \infty$, while that corresponding to $f''_{\rm exp}(0;-0.1) = -0.35026$ goes to the finite asymptotic value $f_{\rm exp}(\infty;-0.1) = +1.76860$.

than the decay of the profile associated with the value $f''_{\rm exp}(0;-0.1)$ lying on the Ingham–Brown curve of Fig. 1, i.e. it only suggests but it does not rigorously prove the algebraic character of their asymptotic behavior. Nevertheless, this conjecture is substantiated once more by Fig. 3 where the plots of the stream functions $f(\eta;\lambda)$ corresponding to the velocity profiles of Fig. 2 are shown. We see in Fig. 3, that all the curves corresponding to $f''(0;-0.1) > f''_{\rm exp}(0;-0.1)$ go to infinity as $\eta \to \infty$, while that corresponding to $f''_{\rm exp}(0;-0.1)$ is the only one which goes to a finite asymptotic value, $f_{\rm exp}(\infty;-0.1) = +1.76860527$.

2.2. The case $\lambda = -1/3$

As it has been shown recently by Magyari et al. [7], the algebraically decaying solutions of the boundary value problem (1) corresponding to the values $f''(0;\lambda) > f''_{\rm exp}(0;\lambda)$ of $f''(0;\lambda)$ can be obtained for $\lambda = -1/3$ in an exact analytic form in terms of the Airy functions,

$$f(\eta;-1/3) = \left[36 f''(0;-1/3)\right]^{1/3} \frac{Bi'(z_0)Ai'(z) - Ai'(z_0)Bi'(z)}{Bi'(z_0)Ai(z) - Ai'(z_0)Bi(z)} \tag{10a}$$

$$f'(\eta;-1/3) = f''(0;-1/3)\cdot\eta + 1 - \frac{1}{6}f^2 \tag{10b}$$

where

$$z = \left[\sqrt{6} f''(0;-1/3)\right]^{-2/3}\left(1 + f''(0;-1/3)\eta\right), \quad z_0 = z|_{\eta=0}. \tag{11}$$

The far field behavior of this solution is

$$f(\eta;-1/3) \to \sqrt{6+6f''(0;-1/3)\eta} \quad \text{as} \quad \eta \to \infty \tag{12a}$$

$$f'(\eta;-1/3) \to 3f''(0;-1/3)\big(6+6f''(0;-1/3)\eta\big)^{-1/2} \quad \text{as} \quad \eta \to \infty \tag{12b}$$

for all $f''(0;-1/3) > f''_{\exp}(0;-1/3) = 0$. Hence, in this special case the values of the exponents present in Eqs. (8) are $b = -1/2$ and $\beta = +1/2$. Now, we actually see that, as already mentioned above, the condition (4) of validity of the integral relationship (3) is not satisfied in this case.

A remarkable feature of the solution (10a) having for $f''(0) > 0$ the algebraic asymptotic behavior (12a) is that for $f''(0) \to 0$ it goes over in the well known hyperbolic tangent solution $f(\eta;-1/3) = \sqrt{6}\tanh\big(\eta/\sqrt{6}\big)$. This property has been proved by Magyari et al. [7] numerically, and later by Magyari and Rees [8] analytically. The analytical proof uses the asymptotic properties of the Airy functions (see e.g. [9]) and yields

$$f(\eta;-1/3) \to \big(6+6f''(0;-1/3)\eta\big)^{-1/2} \cdot \tanh\left(\frac{\big(1+f''(0;-1/3)\eta\big)^{3/2}-1}{(27/2)^{1/2}f''(0;-1/3)}\right) \to \sqrt{6}\tanh\left(\frac{\eta}{\sqrt{6}}\right) \tag{13}$$

as $f''(0;-1/3) \to 0$. In this way for $f''(0;-1/3) = 0$ one obtains $f'(\eta;-1/3) = 1/\cosh^2\big(\eta/\sqrt{6}\big)$ which shows explicitly that the family of solutions $f'(\eta;-1/3)$ having for $f''(0;-1/3) > 0$ the *algebraic* asymptotic decay (12b) goes over for $f''(0;-1/3) \to 0$ in a solution which decays *exponentially*, $f'(\eta;-1/3) \to 4\exp\big(-2\eta/\sqrt{6}\big)$. as $\eta \to \infty$. The main issue of the present paper is to prove this remarkable feature, i.e. the continuous crossover of a family of algebraically decaying boundary layers into an exponentially decaying one, holds for the whole parameter range $-1/2 < \lambda < 0$ of the boundary value problem (1). The numerical "proof" has already been illustrated in Fig. 2. The detailed analytical proof is presented in the Appendix and its main results are summarized in Sect. 2.3 below.

2.3. Series solutions

With the aid of the homotopy analysis method (Liao, [6]) we obtain the following series solution of the problem (1):

$$f(\eta;\lambda) = \frac{1}{\alpha}\sum_{k=0}^{+\infty}\sum_{n=1}^{2k+2}\sum_{m=n-1+\mathcal{X}_n}^{2k+1+[n/2]} A_k^{n,m}(1+\alpha\eta)^{n\beta-m}. \tag{14}$$

Here $[x]$ stands for the integer part of a real number $x, \alpha > 0$, is a constant, and the coefficients $A_k^{m,n}$ can recursively be calculated from the equations

$$A_k^{n,m} = \frac{\hbar\left(\alpha^2 B_k^{n,m} + C_k^{n,m}\right)}{(n\beta - m - \beta)(n\beta - m - \beta + 1)(n\beta - m - 2\beta + 2)} + A_{k-1}^{n,m}\mathcal{X}_{2k+2-n}\mathcal{X}_{2k+1+[n/2]-m} \tag{15}$$

where $\hbar$ is an auxiliary parameter, $n\beta - m \neq \beta, \beta - 1, 2\beta, \beta - 2$ and

$$B_k^{n,m} = \mathcal{X}_{2k+2-n}\mathcal{X}_{2k+3-m+[n/2]}\mathcal{X}_{m+1-n-\mathcal{X}_n}(n\beta - m + 2)(n\beta - m + 1)(n\beta - m)A_{k-1}^{n,m-2}, \tag{16}$$

$$C_k^{m,s} = \sum_{n=0}^{k-1} \sum_{p=\max\{1,m+2n-2k\}}^{\min\{2n+2,m-1\}} \sum_{q=\max\{p-1+\mathcal{X}_p, s+2n-2k-[(m-p)/2]\}}^{\min\{2n+1+[p/2], s+p-m-\mathcal{X}_{m-p}\}} \mathcal{X}_m\mathcal{X}_{2k-s+3+[p/2]+[(m-p)/2]}\mathcal{X}_{s+3-m-\mathcal{X}_p-\mathcal{X}_{m-p}}$$

$$\times[(m-p)\beta + q - s + 1]\left\{\frac{(1+\lambda)}{2}[(m-p)\beta + q - s] - \lambda(p\beta - q)\right\} A_n^{p,q} A_{k-1-n}^{m-p,s-q-1} \tag{17}$$

and

$$\begin{aligned} A_k^{1,0} &= \frac{2(\beta-1)^2\delta_0 + (4-3\beta)\delta_1 + \delta_2}{\beta - 2}, \\ A_k^{1,1} &= -2\beta\delta_0 + 3\delta_1 - \frac{\delta_2}{(\beta - 1)}, \\ A_k^{2,2} &= \frac{\beta(1-\beta)\delta_0 - 2(1-\beta)\delta_1 - \delta_2}{(\beta-1)(\beta-2)}. \end{aligned} \tag{18}$$

In the latter equations,

$$\begin{aligned} \delta_0 &= \sum_{n=1}^{2k+2} \sum_{m=n-1+\mathcal{X}_n}^{2k+1+[n/2]} A_k^{n,m}\mathcal{X}_{(n-1)^2+m^2+1}\mathcal{X}_{(n-1)^2+(m-1)^2+1}\mathcal{X}_{(n-2)^2+(m-2)^2+1}, \\ \delta_1 &= \sum_{n=1}^{2k+2} \sum_{m=n-1+\mathcal{X}_n}^{2k+1+[n/2]} A_k^{n,m}(n\beta - m)\mathcal{X}_{(n-1)^2+m^2+1}\mathcal{X}_{(n-1)^2+(m-1)^2+1}\mathcal{X}_{(n-2)^2+(m-2)^2+1}, \\ \delta_2 &= \sum_{n=1}^{2k+2} \sum_{m=n-1+\mathcal{X}_n}^{2k+1+[n/2]} A_k^{n,m}(n\beta - m)(n\beta - m - 1)\times \\ &\quad \mathcal{X}_{(n-1)^2+m^2+1}\mathcal{X}_{(n-1)^2+(m-1)^2+1}\mathcal{X}_{(n-2)^2+(m-2)^2+1}. \end{aligned} \tag{19}$$

In above expressions,

$$\gamma = f''(0;\lambda) \tag{20}$$

is given (see the existence domain shown in Fig. 1) and $\mathcal{X}_k$ is defined by

$$\mathcal{X}_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1. \end{cases} \tag{21}$$

The first three coefficients are

$$A_0^{1,0}=\frac{4-3\beta+\gamma\alpha^{-1}}{2-\beta},\quad A_0^{1,1}=\frac{3-3\beta+\gamma\alpha^{-1}}{\beta-1},\quad A_0^{2,2}=\frac{2-2\beta+\gamma\alpha^{-1}}{(1-\beta)(2-\beta)}.\tag{22}$$

Further details are presented in the Appendix of the present paper.

In terms of the first three coefficients (22) we can obtain all coefficients of the expansion (14) one by one, both symbolically or numerically. In this way the series solution (14) with algebraic asymptotic decay is fully determined. For example, for the parameter values $\lambda=-1/3,\gamma=1,\alpha=1/2$ and $\hbar=-4$ we obtain:

$$\begin{array}{llll}
A_0^{1,0}=3, & A_0^{1,1}=-7, & A_0^{2,2}=4, & \\
A_1^{1,0}=-\dfrac{3079}{2520}, & A_1^{1,1}=\dfrac{1331}{72}, & A_1^{1,2}=\dfrac{9}{8}, & A_1^{1,3}=\dfrac{35}{24}, \\
A_1^{2,2}=-36, & A_1^{2,3}=\dfrac{784}{45}, & A_1^{2,4}=-\dfrac{48}{35}, & \\
A_1^{3,3}=12, & A_1^{3,4}=-\dfrac{140}{9}, & & \\
A_1^{4,4}=0, & A_1^{4,5}=\dfrac{128}{35}, & & \\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}\tag{23}$$

and thus

$$\begin{aligned}f\left(\eta;-\frac{1}{3}\right)=&\frac{4481}{1260}\left(1+\frac{1}{2}\eta\right)^{\frac{1}{2}}+\frac{827}{36}\left(1+\frac{1}{2}\eta\right)^{-\frac{1}{2}}-64\left(1+\frac{1}{2}\eta\right)^{-1}+\frac{105}{4}\left(1+\frac{1}{2}\eta\right)^{-\frac{3}{2}}\\&+\frac{1568}{45}\left(1+\frac{1}{2}\eta\right)^{-2}-\frac{1015}{36}\left(1+\frac{1}{2}\eta\right)^{-\frac{5}{2}}+\frac{32}{7}\left(1+\frac{1}{2}\eta\right)^{-3}+\ldots\end{aligned}\tag{24}$$

3. Discussion and conclusions

As shown in Fig. 4, for $\lambda=-1/3$ the series solution (14) agrees well with the exact solution (10) already in the 20th-order of approximation, and its convergence can further be accelerated by means of the homotopy–Páde technique (Liao, [6]). This verifies the validity of the homotopy approach applied.

The expression (24) of $f(\eta;-1/3)$ obtained by truncation of (14) yields an accurate approximation for small values of η. It also describes the asymptotic behaviour of $f(\eta;-1/3)$ with an acceptable accuracy. Indeed, for $\eta\to\infty$ the leading order term of (24) behaves as

$$f\left(\eta;-\frac{1}{3}\right)=\sqrt{2}(A_0^{1,0}+A_1^{1,0})\sqrt{\eta}=\sqrt{2}\left(3-\frac{3079}{2520}\right)\sqrt{\eta}=2.514718\cdot\sqrt{\eta}.\tag{25}$$

This fits the asymptotic expression (12a) of the exact solution (10a),

$$f_{exact}\left(\eta;-\frac{1}{3}\right)=\sqrt{6f''(0;-1/3)\eta}=\sqrt{6}\cdot\sqrt{\eta}=2.449489\cdot\sqrt{\eta}\tag{26}$$

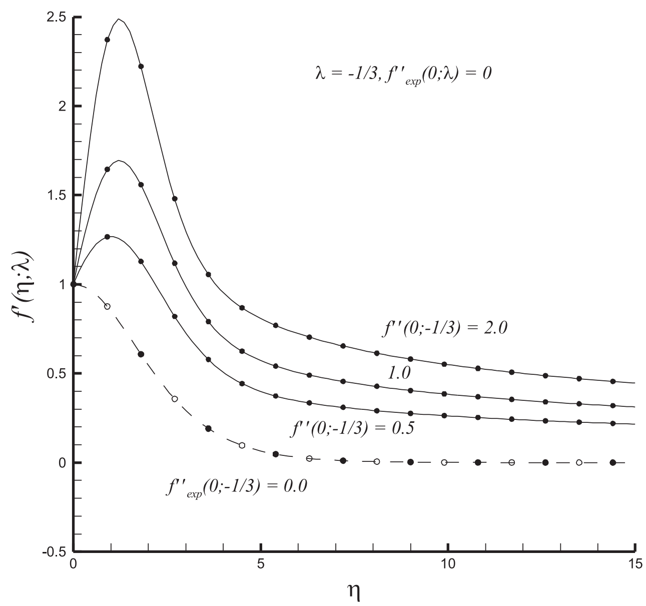

Figure 4. Comparison of the algebraically decaying series solutions (14) (filled circles) associated with different positive values of $f''(0;-1/3)$, with the analytic solution (10) valid for $\lambda = -1/3$ (lines). The dashed line correspond to the well known exponentially decaying solution $f'_{\mathrm{exp}}(\eta;-1/3) = 1/\cosh^2(\eta/\sqrt{6})$ with $f''(0;-1/3) = 0$. The open circles on this curve denote series solution given by Liao and Pop, [5]. As $f''(0;-1/3)$ approaches the value $f''_{\mathrm{exp}}(0;-1/3) = 0$, the family of algebraically decaying profiles goes over continuously in the exponentially decaying one.

with a deviation of only +2.7%.

The accuracy of the leading order term of (24) can obviously be enhanced by taking into account in (14) several terms the degree $(1+\eta/2)^{1/2}$. By doing so, one obtains

$$f\left(\eta;-\frac{1}{3}\right) = \left(\sqrt{2}\cdot\sum_{n=0}^{m} A_n^{1;0}\right)\sqrt{\eta} \quad \text{as} \quad \eta \to \infty. \tag{27}$$

One finds that for $m = 40$ in (27) the coefficient of $\sqrt{\eta}$ in (27) approaches (although not monotonously) the value 2.4496.

In general, for any specified $\lambda \in (-1/2, 0)$ we can get convergent algebraically decaying series solution for any $f''(0;\lambda) > f''_{\mathrm{exp}}(0;\lambda)$ in a similar way by choosing the values of the control parameters α and $\hbar$ suitably. On the other hand, we fail to get any convergent series solutions if $f''(0;\lambda) < f''_{\mathrm{exp}}(0;\lambda)$, in a full agreement with the existence domain shown in Fig. 1. As an illustration, in Figs. 5 and 6 the

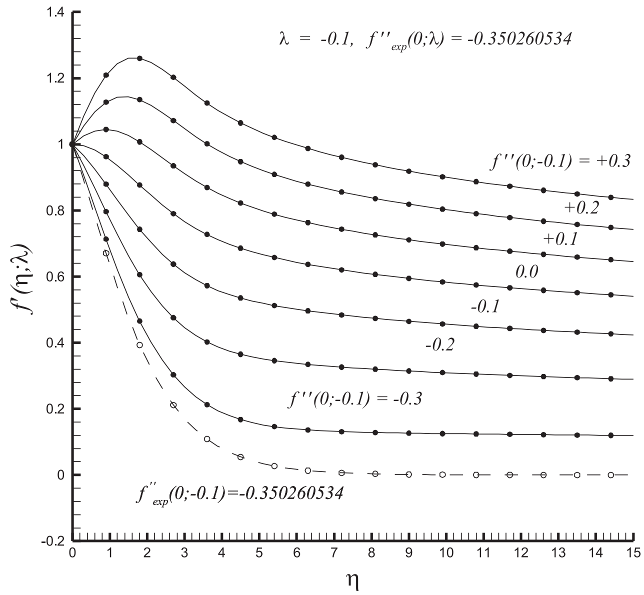

Figure 5. Comparison of the series solution (symbols) with the numerical results (lines) for $\lambda = -1/10$. The dashed and solid lines correspond to the exponentially decaying and the seven algebraically decaying dimensionless temperature (or velocity) profiles $f'_{\text{exp}}(\eta;\lambda)$ and $f'(\eta;\lambda)$, respectively. The open circles denote exponentially decaying series solution given by Liao and Pop, [5], and the filled circles are the algebraically decaying series solution obtained in this article. As $f'(\eta;\lambda)$ approaches the value of $f'_{\text{exp}}(\eta;\lambda)$, the family of algebraically decaying profiles goes over continuously in the exponentially decaying one.

cases $\lambda = -1/10$ and $\lambda = -19/39 \approx -0.4872$ are presented, respectively.

Therefore, the solutions obtained by the homotopy analysis method and presented in Sect. 2.3 (with details in the Appendix) prove the main result of the present paper that in the parameter range $-1/2 < \lambda < 0$ of the Cheng and Minkowycz problem (1) the well known exponentially decaying solutions are not some isolated solutions but limiting cases of families of algebraically decaying multiple solutions. In other words, in the range $-1/2 < \lambda < 0$ the points of the Ingham–Brown curve $f''_{\text{exp}}(0;\lambda)$ shown in Fig. 1 are in fact branching points. From every point of this curve there bifurcates a whole family of algebraically decaying solutions corresponding to values $f''(0;\lambda) > f''_{\text{exp}}(0;\lambda)$ of the dimensionless wall temperature gradient. In addition to the different asymptotic decay of the dimensionless temperature (and velocity) fields $f'(\eta;\lambda)$ associated with the points of the Ingham–Brown curve $f''_{\text{exp}}(0;\lambda)$ on the one hand and with the values of $f''(0;\lambda)$ above it, $f''(0;\lambda) > f''_{\text{exp}}(0;\lambda)$, the corresponding entrainment velocities $f(\infty;\lambda)$

are also basically different, as already predicted numerically (see Fig. 2). Indeed, while $f_{\exp}(\infty;\lambda)$ is a finite quantity, in the case of the bifurcating algebraically decaying solutions $f(\eta;\lambda) \to \infty$ as $\eta \to \infty$. These diverging entrainment velocities are given according to Eq. (14) by

$$f(\eta,\lambda) = \alpha^{\beta-1}\left(\sum_{n=0}^{m} A_n^{1,0}\right) \cdot \eta^{\beta} \quad \text{as} \quad \eta \to \infty. \tag{28}$$

Appendix: Derivation of the series solution (14)

Consider the solutions with algebraic asymptotic property at infinity

$$f \sim a\eta^{\beta}, \quad \eta \to +\infty, \tag{A-1}$$

where β is defined by Eq. (9). When $-1/2 < \lambda < 0$, it holds

$$2\beta - 2 < \beta - 1 < \beta < 1.$$

Under the transformation

$$f(\eta;\lambda) = g(\xi;\lambda)/\alpha, \;\; \xi = 1 + \alpha\eta, \;\; \alpha > 0,$$

the original equation becomes

$$\alpha^2 g'''(\xi;\lambda) + \left(\frac{1+\lambda}{2}\right) g(\xi;\lambda) g''(\xi;\lambda) - \lambda[g'(\xi;\lambda)]^2 = 0 \tag{A-2}$$

subject to the boundary conditions

$$g(1;\lambda) = 0, \quad g'(1;\lambda) = 1, \quad g'(+\infty;\lambda) = 0, \tag{A-3}$$

where the primes denote the differentiation with respect to ξ.

According to the algebraic property at infinity, (A-1), and the boundary conditions (A-3), the solution can be expressed by the following set of base functions

$$\left\{\xi^{m\beta-n} | m\beta - n < 1, \beta < 1, m \in \mathbf{N}, n \in \mathbf{N}\right\}$$

in the form

$$g(\xi;\lambda) = b_{1,0}\xi^{\beta} + \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} b_{m,n}\xi^{m(\beta-n)}, \tag{A-4}$$

which provides us with the so-called Rule of Solution Expression suggested by Liao, [6]. Note that a solution expressed by the Rule of Solution Expression (A-4) automatically satisfies the boundary condition at infinity, i.e. $g'(+\infty;\lambda) = 0$, which therefore becomes a non-effective boundary condition. The problem can be closed by providing an additional boundary condition $f''(0;\lambda) = \gamma$, corresponding to

$$g''(1;\lambda) = \gamma/\alpha, \tag{A-5}$$

where γ is a given constant.

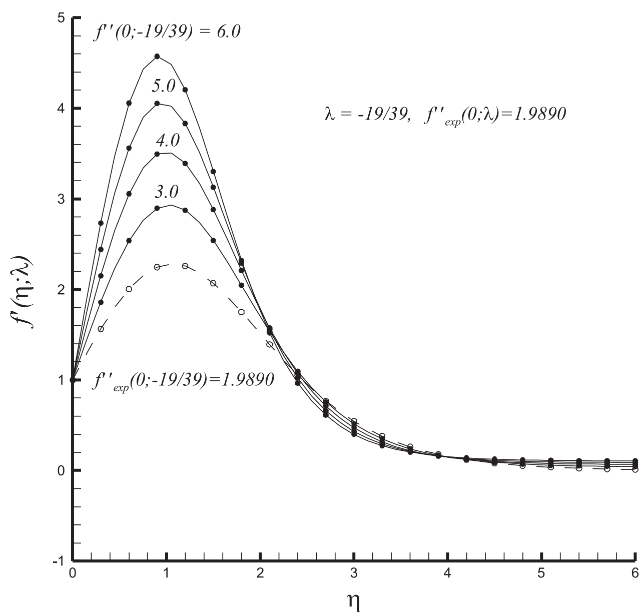

Figure 6. Comparison of the series solution (symbols) with the numerical results (lines) for $\lambda = -19/39$. The dashed and solid lines correspond to the exponentially decaying and the four algebraically decaying dimensionless temperature (or velocity) profiles $f'_{\text{exp}}(\eta;\lambda)$ and $f'(\eta;\lambda)$, respectively. The open circles denote exponentially decaying series solution given by Liao and Pop, [5], and the filled circles are the algebraically decaying series solution obtained in this article. As $f'(\eta;\lambda)$ approaches the value of $f'_{\text{exp}}(\eta;\lambda)$, the family of algebraically decaying profiles goes over continuously in the exponentially decaying one.

According to the Rule of Solution Expression (A-4) and the boundary conditions (A-3) and (A-5), it is straightforward to choose an initial approximation

$$g_0(\xi;\lambda) = \frac{(4-3\beta+\gamma\alpha^{-1})}{(2-\beta)}\xi^{\beta} - \frac{(3-3\beta+\gamma\alpha^{-1})}{(1-\beta)}\xi^{\beta-1} + \frac{(2-2\beta+\gamma\alpha^{-1})}{(1-\beta)(2-\beta)}\xi^{2\beta-2}. \tag{A-6}$$

From Eq. (A-2) and the Rule of Solution Expression (A-4), we choose the auxiliary linear operator

$$\mathbf{L}[f] = \xi^3 f''' + a_2(\xi)\xi^2 f'' + a_1(\xi)\xi f' + a_0(\xi) f, \tag{A-7}$$

where the primes denote the differentiation with respect to $\xi, a_0(\xi), a_1(\xi), a_2(\xi)$ are unknown functions to be determined soon. Enforcing the first three base functions

$$\xi^\beta, \xi^{\beta-1}, \xi^{2\beta-2}$$

to be the solutions of the linear equation $\mathbf{L}[f] = 0$, we have

$$a_0 = -2\beta(1-\beta)^2, \quad a_1 = (\beta-1)(5\beta-6), \quad a_2 = -2(2\beta-3).$$

Thus, to obey the Rule of Solution Expression (A-4), it is natural for us to choose the linear auxiliary operator

$$\mathbf{L}[f] = \xi^3 f''' - 2(2\beta-3)\xi^2 f'' + (\beta-1)(5\beta-6)\xi f' - 2\beta(\beta-1)^2 f, \tag{A-8}$$

which possesses the property

$$\mathbf{L}\left[C_0\xi^\beta + C_1\xi^{\beta-1} + C_2\xi^{2\beta-2}\right] = 0, \tag{A-9}$$

for any constants C_0, C_1 and C_2. Besides, we are led from Eq. (A-2) to define a nonlinear operator

$$\begin{aligned} \mathbf{N}[\phi(\xi;\lambda,q)] = \alpha^2 \frac{\partial^3\phi(\xi;\lambda,q)}{\partial\xi^3} + \left(\frac{1+\lambda}{2}\right)\phi(\xi;\lambda,q)\frac{\partial^2\Phi(\xi;\lambda,q)}{\partial\xi^2} \\ - \lambda\left[\frac{\partial\phi(\xi;\lambda,q)}{\partial\xi}\right]^2, \end{aligned} \tag{A-10}$$

where $q \in [0,1]$ is an embedding parameter. Let $\hbar$ denote a non-zero auxiliary parameter, $\mathbf{H}(\xi)$ a non-zero auxiliary function, and $q \in [0,1]$ is an embedding parameter. We construct the so-called zeroth-order deformation equation

$$(1-q)\mathbf{L}[\phi(\xi;\lambda,q) - g_0(\xi;\lambda)] = q\hbar\mathbf{H}(\xi)\mathbf{N}[\phi(\xi;\lambda,q)], \tag{A-11}$$

subject to the boundary conditions

$$\begin{aligned} \phi(1;\lambda,q) = 0, \quad \left.\frac{\partial\phi(\xi;\lambda,q)}{\partial\xi}\right|_{\xi=1} = 1, \quad \left.\frac{\partial^2\phi(\xi;\lambda,q)}{\partial\xi^2}\right|_{\xi=1} \\ = \frac{\gamma}{\alpha}, \quad \left.\frac{\partial\phi(\xi;\lambda,q)}{\partial\xi}\right|_{\xi\to+\infty} = 0. \end{aligned} \tag{A-12}$$

Obviously, when $q = 0$, we have from Eqs. (A-11) and (A-12) the solution

$$\phi(\xi;\lambda,0) = g_0(\xi;\lambda). \tag{A-13}$$

When $q = 1$, Eqs. (A-11) and (A-12) are equivalent to the original ones (A-2), (A-3) and (A-5), provided

$$\phi(\xi;\lambda,1) = g(\xi;\lambda). \tag{A-14}$$

Thus, as q increases from 0 to 1, the solution $\phi(\xi;\lambda,q)$ of the zeroth-order deformation equations (A-11) and (A-12) varies from the initial approximation $g_0(\xi;\lambda)$ to the solution of the original equations (A-2), (A-3) and (A-5).

Assume that $\hbar$ and $\mathbf{H}(\xi)$ are properly chosen so that the variation (or deformation) is smooth enough and thus $\phi(\xi;\lambda,q)$ can be expanded in the Taylor series

$$\phi(\xi;\lambda,q)=\phi(\xi;\lambda,0)+\sum_{k=1}^{+\infty}g_k(\xi;\lambda)q^k, \tag{A-15}$$

where

$$g_k(\xi;\lambda)=\frac{1}{k!}\frac{\partial+k\phi(\xi;\lambda,q)}{\partial q^k}\bigg|_{q=0},$$

and besides the series (A-15) converges at $q=1$. Then, using (A-13) and (A-14), we have the solution series

$$g(\xi;\lambda)=g_0(\xi;\lambda)+\sum_{k=1}^{+\infty}g_k(\xi;\lambda). \tag{A-16}$$

which provides us with a relationship between the solution $g(\xi;\lambda)$ and the initial guess $g_0(\xi;\lambda)$.

Define the vector

$$\overrightarrow{g}_k=\left\{g_0(\xi;\lambda),g_1(\xi;\lambda),g_2(\xi;\lambda),\ldots,g_k(\xi,\lambda)\right\}.$$

To obtain governing equation and boundary conditions for the unknown $g_k(\xi;\lambda)$ in the order $k=1,2,3,\ldots$, we differentiate the zeroth-order deformations (A-11) and (A-12) k times with respect to the embedding parameter q, then divide them by $k!$, and finally set $q=0$. In this way, we have the so-called k th-order deformation equation

$$\mathbf{L}[g_k(\xi;\lambda)-\mathcal{X}_k g_{k-1}(\xi;\lambda)]=\hbar\mathbf{H}(\xi)R_k(\overrightarrow{g}_{k-1}), \tag{A-17}$$

subject to the boundary conditions

$$g_k(1;\lambda)=g_k'(1;\lambda)=g_k''(1;\lambda)=g_k'(+\infty;\lambda)=0, \tag{A-18}$$

where

$$R_k(\overrightarrow{g}_{k-1})=\alpha^2(g_{k-1})'''+\sum_{n=0}^{k-1}\left[\left(\frac{1+\lambda}{2}\right)g_n(g_{k-1-n})''-\lambda(g_n)'(g_{k-1-n})'\right] \tag{A-19}$$

and

$$\mathcal{X}_k=\begin{cases}0, & k\le 1,\\ 1, & k>1.\end{cases} \tag{A-20}$$

Note that, substituting (A-15) in Eqs. (A-11) and (A-12), and equating the coefficients of the same powers of q, we can obtain the same equations as given above.

To obey the Rule of Solution Expression (A-4), the auxiliary function $\mathbf{H}(\xi)$ must be in the form $\mathbf{H}(\xi)=\xi^\mu$, where μ is an integer to be determined. It is found that, when $\mu\ge 2, R_k(\overrightarrow{g}_{k-1})$ contains the term $\xi^{2\beta-2}$, and thus, due to (A-9), the solution $g_k(\xi)$ has the term $\xi 1n\xi$. However, the term $\xi 1n\xi$ disobeys the

Rule of Solution Expression (A-4). When $\mu \leq 0$, the solutions of the high-order deformation equations do not contain the term $\xi^{2\beta-1}$. This, however, disobeys the Rule of Coefficient Ergodicity, i.e. coefficients of all base functions could be modified as the order of approximation tends to infinity, as suggested by Liao (2003). Thus, we must choose $\mu = 1$, corresponding to $\mathbf{H}(\xi) = \xi$. In this way, all of the linear equations (A-18) and (A-19) have solutions which comply the Rule of Solution Expression (A-4). And it is found that $g_k(\xi;\lambda)$ can be expressed in such a general form

$$g_k(\xi,\lambda) = \sum_{n=1}^{2k+2} \sum_{m=n-1+\mathcal{X}_n}^{2k+1+[n/2]} A_k^{n,m} \xi^{n\beta-m}, \ k = 0, 1, 2, 3, \cdots \tag{A-21}$$

where $A_s^{n,m}$ is a constant coefficient, the operator $[x]$ takes the integer part of the number x. Substituting (A-21) into (A-19), we have

$$R_k = \sum_{n=1}^{2k+2} \sum_{m=n-1+\mathcal{X}_n}^{2k+1+[n/2]} \left(\alpha^2 B_k^{n,m} + C_k^{n,m}\right) \xi^{n\beta-m-1},$$

where

$$B_k^{n,m} = \mathcal{X}_{2k+2-n} \mathcal{X}_{2k+3-m+[n/2]} \mathcal{X}_{m+1-n-\mathcal{X}_n} (n\beta-m+2)(n\beta-m+1)(n\beta-m) A_{k-1}^{n,m-2},$$

and

$$C_k^{m,s} = \sum_{n=0}^{k-1} \sum_{p=\max\{1,m+2n-2k\}}^{\min\{2n+2,m-1\}} \sum_{q=\max\{p-1+\mathcal{X}_p,s+2n-2k-[(m-p)/2]\}}^{\min\{2n+1+[p/2],s+p-m-\mathcal{X}_{m-p}\}} \mathcal{X}_m \mathcal{X}_{2k-s+3+[p/2]+[(m-p)/2]} \mathcal{X}_{s+3-m-\mathcal{X}_p-\mathcal{X}_{m-p}}$$
$$\times [(m-p)\beta + q - s + 1] \left\{ \frac{1+\lambda}{2} [(m-p)\beta + q - s] - \lambda(p\beta - q) \right\} A_n^{p,q} A_{k-1-n}^{m-p,s-q-1}.$$

Substituting these expressions into the high-order deformation equations (A-18) and (A-19), we obtain the recurrence formulas given in section 3.2. The first three coefficients $A_0^{1,0}$, $A_0^{1,1}$, $A_0^{2,2}$ are obtained by comparing (A-21) with the initial guess (A-6).

The above recurrence formulas contain two auxiliary parameters, $\alpha > 0$ and $\hbar$. The value of $\alpha > 0$ is determined by the minimum value of the residual error of the governing equation about the initial guess (A-6). Then, only the auxiliary parameter $\hbar$ is unknown, which provides us with a simple way to control and adjust the convergence of the series (A-16), as shown by Liao, [5]. In all cases considered in this articles, we choose $\alpha = 1/2$ and $\hbar = -4$.

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(Received: June 7, 2005)

Published Online First: June 15, 2006