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A challenging nonlinear problem for numerical techniques

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Abstract

In this paper we show that a nonlinear boundary-value problem describing Blasius viscous flow of a kind of non-Newtonian fluid has an infinite number of explicit analytic solutions. These solutions are rather sensitive to the second-order derivative at the boundary, and the difference of the second derivatives of two obviously different solutions might be less than 10^{-1000}. Therefore, it seems impossible to find out all of these solutions by means of current numerical methods. Thus, this nonlinear problem might become a challenge to current numerical techniques.

MSC: 34B15; 68Q17; 65L12; 65L17

Keywords: Multiple solution; Nonlinearity; Sensitivity to boundary conditions

1. Introduction

The two-dimensional Blasius viscous flow of power-law fluid on a semi-infinite flat plane [1–5] can be described by the boundary layer equation

$$f'''(f'')^{n-1} + ff'' = 0, \tag{1}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{2}$$

where the prime denotes the differentiation with respect to the simplicity variable η. The numerical results of $f''(0)$ when $n = 1$ and $n = 2$ are given by Kim et al. [4] and Akcay et al. [5], as shown in Table 1.

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Table 1
Numerical results of $f''(0)$ when $n=1$ and $n=2$

n	Value given by Kim et al. [4]	Value given by Akcay et al. [5]
1	0.469599988	0.4696
2	0.726468	0.7274

The discontinuities in this equation were first investigated by Teipel [6]. He pointed out the difficulty to satisfy the outer boundary condition, not only when the power-law index is equal to 2 ($n=2$), but also when $1<n<2$ and $n>2$. He indicated to the discontinuities in the third and fourth derivatives of the function f for $n>1$. Teipel [6] concluded also that the difficulties encountered are the result of the used power-law model.

2. An infinite number of solutions when $n = 2$

Let us consider a special non-Newtonian fluid, say, $n=2$. In this case, Eq. (1) becomes

$$(f''' + f)f'' = 0, \tag{3}$$

subject to the same boundary conditions (2). The above equation gives either

$$f'''(\eta) + f(\eta) = 0 \tag{4}$$

or

$$f''(\eta) = 0. \tag{5}$$

The solution of Eq. (4) can be generally expressed by

$$f_A(\eta) = C_1\ \exp(-\eta) + \exp(\eta/2)\left[C_2\ \sin\left(\frac{\sqrt{3}}{2}\eta\right) + C_3\ \cos\left(\frac{\sqrt{3}}{2}\eta\right)\right], \tag{6}$$

where C_1, C_2, and C_3 are coefficients to be determined. Substituting the wall boundary conditions

$$f(0) = f'(0) = 0$$

into (6), we have the solution

$$f_A(\eta) = cg(\eta), \tag{7}$$

where c is a coefficient to be determined and the function $g(\eta)$ is defined by

$$g(\eta) = \exp(-\eta) - \exp(\eta/2)\cos\left(\frac{\sqrt{3}}{2}\eta\right) + \sqrt{3}\exp(\eta/2)\sin\left(\frac{\sqrt{3}}{2}\eta\right). \tag{8}$$

Obviously, the above solution does not satisfy the mainstream boundary condition

$$f'(+\infty) = 1.$$

The solution of (5) is given by

$$f_B(\eta) = C_4\eta + d, \tag{9}$$

where C_4 and d are coefficients. Using the mainstream boundary condition

$$f'(+\infty) = 1,$$

we have $C_4 = 1$, therefore

$$f_B(\eta) = \eta + d, \tag{10}$$

which, however, does not satisfy the wall boundary conditions

$$f(0) = f'(0) = 0.$$

So, neither $f_A(\eta)$ nor $f_B(\eta)$ satisfies all of the boundary conditions (2).

Note that $f_A(\eta)$ that satisfies the wall boundary condition can be regarded as an inner solution, and $f_B(\eta)$ that satisfies the mainstream boundary condition at infinity can be seen as an outer solution, provided there exists such a marching point at $\eta = \eta^*$ that

$$f_A(\eta^*) = f_B(\eta^*), \tag{11}$$

$$f'_A(\eta^*) = f'_B(\eta^*), \tag{12}$$

$$f''_A(\eta^*) = f''_B(\eta^*). \tag{13}$$

Substituting (7) and (10) into the above expressions, we have

$$cg(\eta)^* = \eta^* + d, \tag{14}$$

$$cg'(\eta^*) = 1, \tag{15}$$

$$g''(\eta^*) = 0, \tag{16}$$

which determine the unknown coefficients c, d and the matching-point η^*. Due to (16) and (8), the position of the marching-points is given by the nonlinear algebraic equation

$$\exp(-\eta^*) + 2\,\exp(\eta^*/2)\cos\left(\frac{\sqrt{3}}{2}\eta^*\right) = 0. \tag{17}$$

The above equation can be rewritten as

$$\exp\left(-\frac{3}{2}\eta^*\right) = -2\,\cos\left(\frac{\sqrt{3}}{2}\eta^*\right). \tag{18}$$

Obviously, the curve

$$\zeta = \exp\left(-\frac{3}{2}\,\eta^*\right)$$

and the curve

$$\zeta = -2\,\cos\left(\frac{\sqrt{3}}{2}\eta^*\right)$$

have an infinite number of intersections. Thus, Eq. (18) has an infinite number of solutions

$$\begin{aligned}
\eta_1^* &= 1.849812799190,\\
\eta_2^* &= 5.441233355024,\\
\eta_3^* &= 9.068997534872,\\
\eta_4^* &= 12.696595546547,\\
\eta_5^* &= 16.324194278121,\\
&\vdots
\end{aligned}$$

Note that

$$\zeta = \exp\left(-\tfrac{3}{2}\,\eta^*\right)$$

exponentially tends to zero. When $\eta > \eta_5^*$,

$$\exp\left(-\tfrac{3}{2}\,\eta\right) < 2.32 \times 10^{-11},$$

so that the term on the left-hand side of (18) can be approximately regarded as zero and thus

$$\eta_k^* = \frac{(2k-1)\pi}{\sqrt{3}}, \quad k \geqslant 6, \tag{19}$$

is a rather accurate solution of (18) for $k \geqslant 6$.

As long as the position η_k^* of the kth marching-point is known, we have by (15) and (14) the coefficients

$$c_k = \frac{1}{g'(\eta_k^*)}, \tag{20}$$

$$d_k = \frac{g(\eta_k^*)}{g'(\eta_k^*)} - \eta_k^*. \tag{21}$$

Then, the kth solution is given by

$$f_k(\eta) = \begin{cases} c_k\, g(\eta) & \text{when } 0 \leqslant \eta < \eta_k^*, \\ \eta + d_k & \text{when } \eta \geqslant \eta_k^*, \end{cases} \tag{22}$$

where $k = 1, 2, 3, \ldots$. Mathematically, the function defined above satisfies the governing equation (3) in the whole region $0 \leqslant \eta < +\infty$ and all of the boundary conditions (2) at $\eta = 0$ and at infinity. The first 10th solutions are as shown in Fig. 1. Note that the third-order derivatives at the matching point are discontinuous. Teipel [6] also pointed out this kind of discontinuation.

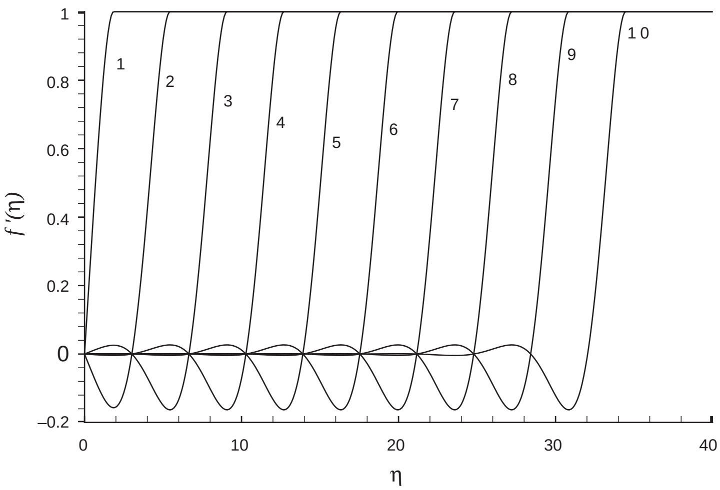

Fig. 1. The curve $f'(\eta)$ of the first 10th solutions.

Table 2
Theoretical value of $f''(0)$ of the kth solution

k	$f_k''(0)$
1	0.726467684935
3	$1.858888323369 \times 10^{-2}$
5	$4.940909271256 \times 10^{-4}$
10	$-5.691039543711 \times 10^{-8}$
50	$-1.763069354421 \times 10^{-39}$
100	$-7.246305476685 \times 10^{-79}$
200	$-1.224083719056 \times 10^{-157}$
500	$-5.900578217031 \times 10^{-394}$
1000	$-8.116466274166 \times 10^{-788}$
2000	$-1.535719227834 \times 10^{-1575}$
5000	$-1.040270594067 \times 10^{-3938}$
10 000	$-2.522728359963 \times 10^{-7877}$

Due to (22) and (8), we have the value of $f''(0)$ of the kth solution, i.e.

$$f_k''(0) = 3c_k. \tag{23}$$

The values of $f''(0)$ of some solutions are listed in Table 2. Note that $f''(0)$ given by the first solution $f_1(\eta)$ agrees with the numerical ones given by Kim et al. [4]. Define

$$\delta_k = \frac{f_k''(0) - f_{k-1}''(0)}{f_{k-1}''(0) - f_{k-2}''(0)}, \quad k \geqslant 3.$$

By means of (19), we have

$$\lim_{k\to+\infty} \delta_k = -\exp\left(-\frac{\pi}{\sqrt{3}}\right) \approx -0.1630335348.$$

3. A challenge to numerical techniques

From pure mathematical points of view, (3) and (2) have an infinite number of solutions expressed by (22). So, Eqs. (2) and (3) provide us with an example that has an infinite number of analytic solutions. Note that, as shown in Table 2, the absolute value of $f''(0)$ of the kth solution decreases so rapidly with respect to k that it becomes more and more difficult to numerically find out $f_k(\eta)$. Note that the value $|f''(0)|$ of the 10 000th solution is equal to $2.522728359963 \times 10^{-7877}$, which is so small that current digital computers cannot even express such a numerical result at such a high accuracy. Although, the difference between $f''_{10+i}(0)$ and $f''_{10+j}(0)$ is less than 6×10^{-8} for any positive integers i and j $(i \neq j)$, the two corresponding solutions $f_{10+i}(\eta)$ and $f_{10+j}(\eta)$ are quite different as η is large enough. So, the solutions of the nonlinear problem under consideration are very *sensitive* to $f''(0)$, and this sensitivity greatly increases the difficulties to numerically find all solutions. It is an open question if our current numerical techniques can find out *all* of these solutions. So, it is a challenge for current numerical and analytic methods.

The first analytic solution $f_1(\eta)$ agrees well with the numerical ones given by Kim et al. [4] and can be observed in nature or in a lab. So, we are quite sure that the first solution $f_1(\eta)$ is physically correct. Although it seems that the solutions with $f''(0) < 0$ have no physical meaning, from mathematical points of view, this problem would be a challenge to current numerical techniques.

Note that $n = 2$ is a special case of Eq. (1). It might be possible that there exist an infinite number of solutions when $n = 3, 4, 5, \ldots$. If so, how can we find out *all* of these solutions by numerical and/or analytic techniques?

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