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International Journal of Non-Linear Mechanics 39 (2004) 271–280
INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS
www.elsevier.com/locate/nlm

An analytic approximate approach for free oscillations of self-excited systems

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Received 31 March 2002

Abstract

A new analytic approximate technique for non-linear problems, namely the homotopy analysis method, is employed to propose an approach for free oscillations of self-excited systems. Different from perturbation methods on this topic, this approach does not depend upon any small/large parameters at all and therefore is valid for free oscillations of all self-excited systems. Besides, unlike other analytic techniques, this approach provides us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. Two examples are employed to illustrate the validity and flexibility of this approach.

Keywords: Non-linear oscillations; Limit-cycle; Self-excited systems

1. Introduction

Perturbation method [1–11] is one of the most widely applied analytic tools for non-linear problems. By means of perturbation techniques, a lot of important properties and interesting phenomena of non-linear problems have been revealed. Recently, the singular perturbation techniques are considered among the top 10 progress of the theoretical and applied mechanics in the 20th century [12]. So, it is out of question that perturbation techniques play important roles in the development of science and engineering.

Essentially, perturbation techniques are based on the existence of a small/large parameter or variable, which is often called *perturbation quantity*. Obviously, the existence of perturbation quantities is a cornerstone of perturbation techniques. However, it is perturbation quantity which brings perturbation techniques some restrictions. First, it is impossible that *each* non-linear problem contains such a perturbation quantity. Secondly, even if perturbation quantities exist, perturbation techniques might fail to give a satisfactory result. For example, both straightforward perturbation method and singular perturbation technique failed to give a satisfactory theoretical drag formula of a sphere in a uniform stream [13,14]. This is mainly because, like other analytic methods, perturbation techniques *themselves* cannot provide us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary.

The dependence of perturbation techniques on small/large parameters might be avoided by introducing a so-called artificial small parameter. In 1892,

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PII: S0020-7462(02)00174-9

Lyapunov [15] considered the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x,$$

where $A(t)$ is a time periodic matrix. Lyapunov [15] introduced an artificial parameter m to replace this equation by the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = mA(t)x$$

and then calculated power series expansions over m for the solutions. In many cases Lyapunov proved that series converge for $m = 1$, and therefore one can put in the final expression by setting $m = 1$. The above approach is called Lyapunov's artificial small parameter method [15]. This idea was further employed by some scientists such as Karmishin et al. [16], who developed the so-called δ-expansion method. However, both the artificial parameter method and δ-expansion method need a principal *rule* to determine the place where the artificial parameter or δ should appear. To the best of our knowledge, such a kind of rule has not been reported. Besides, like perturbation techniques, both the artificial small parameter method and the δ-expansion method *themselves* do not provide us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary.

The idea of artificial parameter can be generalized by the homotopy [17], a concept of topology [18]. Based on the homotopy, many numerical techniques, such as the continuation method [19] and the homotopy continuation method [20], were developed.

In summary, to the best of the author's knowledge, neither perturbation techniques nor artificial small parameter method, nor δ-expansion method provide us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary. Besides, the efficiency of approximating a non-linear problem has not been taken into enough account. So, it is necessary to develop some new, more efficient analytic methods.

A kind of analytic method, namely the homotopy analysis method [21–28,13], was proposed by introducing an auxiliary parameter $\hbar$ to construct a new kind of homotopy in a more general form. Unlike *all* other analytic techniques, the homotopy analysis method always gives a *family* of analytic results at *any* given orders of approximation. The homotopy analysis method has the following advantages:

1. it is valid even if a given non-linear problem does *not* contain any small/large parameters *at all*;
2. it *itself* can provide us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary;
3. it can be employed to *efficiently* approximate a non-linear problem by *choosing* different sets of base functions.

In this paper, let us consider free oscillations of self-excited systems, governed by

$$\ddot{U}(t) = f[U(t), \dot{U}(t), \ddot{U}(t)], \quad t \geqslant 0, \tag{1}$$

where the dot denotes the derivative with respect to the time t.

Free oscillations of self-excited systems have limit-cycles which are independent upon initial conditions. In general, limit cycles of self-excited systems contain two important physical parameters, i.e. the frequency ω and the amplitude a of oscillation. So, without loss of any generality, consider such initial conditions:

$$U(0) = a, \quad \dot{U}(0) = 0, \tag{2}$$

where a is the amplitude of the limit-cycle.

Note that it is *unnecessary* to assume the existence of any small or large parameters in Eq. (1). Thus, the proposed approach is rather general.

2. Basic ideas

Let $\tau = \omega t$ denote a new time scale. Under the transformation

$$\tau = \omega t, \quad U(t) = au(\tau), \tag{3}$$

the original Eq. (1) and its initial condition (2) become

$$\omega^2 u''(\tau) = F[u, u', u'', \omega, a] \tag{4}$$

and

$$u(0) = 1, \quad u'(0) = 0, \tag{5}$$

respectively, where the prime denotes the derivative with respect to τ and

$$F[u,u',u'',\omega,a]=\frac{f[au(\tau),a\omega u'(\tau),a\omega^2u''(\tau)]}{a}. \quad (6)$$

The limit-cycles of self-excited oscillation systems are periodic motions with the period $T=2\pi/\omega$ and thus $u(\tau)$ can be expressed by such a set of base functions

$$\{\sin(m\tau),\cos(m\tau)\,|\,m=0,1,2,3,\ldots\} \quad (7)$$

that

$$u(\tau)=\sum_{k=0}^{+\infty}(\alpha_k\sin k\tau+\beta_k\cos k\tau), \quad (8)$$

where α_k,β_k are coefficients. This provides us with a rule, called *the rule of solution expression*. This rule is important in the frame of the homotopy analysis method, as shown later.

Let ω_0,a_0 denote the initial approximations of the frequency ω and the amplitude a, respectively. Considering the initial conditions (5) and the *rule of solution expression* described by (8), it is obvious that

$$u_0(\tau)=\cos(\tau) \quad (9)$$

is a good initial guess of $u(\tau)$. The homotopy analysis method is based on such continuous variations $\phi(\tau,p),\Omega(p),A(p)$ that, as the embedding parameter p increases from 0 to 1, $\phi(\tau,p)$ varies from the initial guess $u_0(\tau)=\cos\tau$ to the exact solution $u(\tau)$, so does $\Omega(p)$ from ω_0 to the exact frequency ω and $A(p)$ from a_0 to the exact amplitude a, respectively. To ensure this under *the rule of solution expression* described by (8), one chooses such an auxiliary linear operator

$$\mathscr{L}[\phi(\tau,p)]=\omega_0^2\left[\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+\phi(\tau,p)\right] \quad (10)$$

that

$$\mathscr{L}(C_1\sin\tau+C_2\cos\tau)=0, \quad (11)$$

where C_1 and C_2 are coefficients. Note that the *rule of solution expression* described by (8) plays an important role while determining the initial guess and the auxiliary linear operator $\mathscr{L}$. Then, due to (4), one defines the non-linear operator

$$\begin{aligned}\mathscr{N}[\phi(\tau,p),\Omega(p),A(p)]&=\Omega^2(p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}-F\left[\phi(\tau,p),\frac{\partial\phi(\tau,p)}{\partial\tau},\right.\\&\left.\frac{\partial^2\phi(\tau,p)}{\partial\tau^2},\Omega(p),A(p)\right]. \quad (12)\end{aligned}$$

Let $p\in[0,1]$ be the embedding parameter and $\hbar$ a non-zero auxiliary parameter. One constructs such a homotopy in a more general form

$$\begin{aligned}\mathscr{H}[\phi(\tau,p);\hbar,p]&=(1-p)\mathscr{L}[\phi(\tau,p)-u_0(\tau)]\\&\quad-\hbar p\mathscr{N}[\phi(\tau,p),\Omega(p),A(p)]. \quad (13)\end{aligned}$$

Setting $\mathscr{H}[\phi(\tau,p);\hbar,p]=0$, one has a family of equations

$$\begin{aligned}&(1-p)\mathscr{L}[\phi(\tau,p)-u_0(\tau)]\\&\quad=\hbar p\mathscr{N}[\phi(\tau,p),\Omega(p),A(p)], \quad (14)\end{aligned}$$

subject to the initial conditions,

$$\phi(0,p)=1,\qquad \left.\frac{\partial\phi(\tau,p)}{\partial\tau}\right|_{\tau=0}=0. \quad (15)$$

Obviously, when $p=0$, Eqs. (14) and (15) have the solution

$$\phi(\tau,0)=u_0(\tau)=\cos\tau. \quad (16)$$

When $p=1$, Eqs. (14) and (15) are exactly the same as Eqs. (4) and (5), provided

$$\phi(\tau,1)=u(\tau), \quad (17)$$

$$\Omega(1)=\omega, \quad (18)$$

$$A(1)=a. \quad (19)$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi(\tau,p)$ varies from the initial guess $u_0(\tau)=\cos\tau$ to the solution $u(\tau)$, so does $\Omega(p)$ from its initial guess ω_0 to the exact frequency ω, and $A(p)$ from its initial guess a_0 to the exact amplitude a of the limit cycle, respectively. For brevity, call (14) and (15) the zero-order deformation equation.

Assume that $\phi(\tau,p),\Omega(p),A(p)$ are analytic in $p\in[0,1]$ so that the so-called deformation derivatives:

$$u_0^{[k]}(\tau)=\left.\frac{\partial^k\phi(\tau,p)}{\partial p^k}\right|_{p=0},\quad \omega_0^{[k]}=\left.\frac{d^k\Omega(p)}{dp^k}\right|_{p=0},$$

$$a_0^{[k]}=\left.\frac{d^kA(p)}{dp^k}\right|_{p=0} \tag{20}$$

exist. Then, $\phi(\tau,p),\Omega(p),A(p)$ can be expanded in the Maclaurin series of p as follows:

$$\phi(\tau,p)=\sum_{k=0}^{+\infty}u_k(\tau)p^k, \tag{21}$$

$$\Omega(p)=\sum_{k=0}^{+\infty}\omega_k p^k, \tag{22}$$

$$A(p)=\sum_{k=0}^{+\infty}a_k p^k, \tag{23}$$

where

$$u_k(\tau)=\frac{1}{k!}\left.\frac{\partial^k\phi(\tau,p)}{\partial p^k}\right|_{p=0},\quad \omega_k=\frac{1}{k!}\left.\frac{d^k\Omega(p)}{dp^k}\right|_{p=0},$$

$$a_k=\frac{1}{k!}\left.\frac{d^kA(p)}{dp^k}\right|_{p=0}. \tag{24}$$

Notice that series (21)–(23) contain the auxiliary parameter $\hbar$, which determines their convergence regions. Assume that $\hbar$ is properly chosen such that all of these Maclaurin series are convergent at $p=1$. Thus, due to (17)–(19), one has at $p=1$ that

$$u(\tau)=\sum_{k=0}^{+\infty}u_k(\tau), \tag{25}$$

$$\omega=\sum_{k=0}^{+\infty}\omega_k, \tag{26}$$

$$a=\sum_{k=0}^{+\infty}a_k. \tag{27}$$

The results at the mth-order approximation are given by

$$u(\tau)\approx\sum_{k=0}^{m}u_k(\tau), \tag{28}$$

$$\omega\approx\sum_{k=0}^{m}\omega_k, \tag{29}$$

$$a\approx\sum_{k=0}^{m}a_k. \tag{30}$$

Differentiating k times Eqs. (14) and (15) with respect to p and then setting $p=0$ and finally dividing them by $k!$, one gains the governing equation of $u_k(\tau)$, i.e.

$$\mathscr{L}[u_k(\tau)-\chi_k u_{k-1}(\tau)]=\hbar R_k(\tau), \tag{31}$$

subject to the initial conditions

$$u_k(0)=0,\quad u_k'(0)=0, \tag{32}$$

where

$$R_k(\tau)=\frac{1}{(k-1)!}\left.\frac{\partial^{k-1}\mathscr{N}[\phi(\tau,p),\Omega(p),A(p)]}{\partial p^{k-1}}\right|_{p=0} \tag{33}$$

and

$$\chi_k=\begin{cases}0, & k\leqslant 1,\\ 1, & k>1.\end{cases} \tag{34}$$

Notice that both ω_k and a_k remain unknown right now. Due to *the rule of solution expression* described by (8) and the definition (10) of $\mathscr{L}$, solutions of (31) and (32) should *not* contain the so-called secular terms $\tau\sin(\tau)$ and $\tau\cos(\tau)$. To ensure so, the right-hand side term $R_k(\tau)$ of (31) should not contain the terms $\sin\tau$ and $\cos\tau$, i.e. the coefficients of $\sin\tau$ and $\cos\tau$ must be zero. So, rewrite

$$R_k(\tau)=\sum_{n=1}^{\varphi(k)}[c_{k,n}\cos(n\tau)+d_{k,n}\sin(n\tau)], \tag{35}$$

where the integer $\varphi(k)$ is a function of k and the coefficients

$$c_{k,n}=\frac{2}{\pi}\int_0^{\pi}R_k(\tau)\cos(n\tau)\,d\tau,$$

$$d_{k,n}=\frac{2}{\pi}\int_0^{\pi}R_k(\tau)\sin(n\tau)\,d\tau$$

become zero when $n > \varphi(k)$. Then, one gains two algebraic equations

$$c_{k,1}(\omega_0, \omega_1, \ldots, \omega_{k-1}, a_0, a_1, \ldots, a_{k-1}) = 0 \tag{36}$$

and

$$d_{k,1}(\omega_0, \omega_1, \ldots, \omega_{k-1}, a_0, a_1, \ldots, a_{k-1}) = 0, \tag{37}$$

which determine ω_{k-1} and a_{k-1} $(k = 1, 2, 3, \ldots)$. The above two algebraic equations are often non-linear for ω_0 and a_0 when $k = 1$, but always linear otherwise, as proved by Liao [21]. Notice that when $k \geqslant 1$ both (31) and (32) are always *linear*. So, after solving ω_{k-1} and a_{k-1}, it is easy to gain its solution

$$u_k(\tau) = \chi_k u_{k-1}(\tau) + \sum_{n=2}^{\varphi(k)} \frac{c_{k,n}\cos(n\tau) + d_{k,n}\sin(n\tau)}{\omega_0^2(1-n^2)} + C_1\cos\tau + C_2\sin\tau, \tag{38}$$

where the two integral coefficients are determined by initial conditions (32). In this way, one can gain ω_{k-1}, a_{k-1} and $u_k(\tau)$ $(k = 1, 2, 3, \ldots)$, successively.

Note that, to gain uniformly valid approximations, some perturbation techniques were developed to avoid the appearance of secular terms in perturbation solutions. This kind of technique goes back to some scientists in the 19th century, such as Lindstedt [29], Bohlin [30], Poincaré [31], Gyldén [32] and so on. The idea was further developed by Lighthill [33,34], Malkin [35], Kuo [36,37], Tsien [38] and so on. *The rule of solution expression* can be seen as a kind of generalization of this idea. It is under this rule that the auxiliary linear operator $\mathscr{L}$ defined by (10) and the initial guess $u_0(\tau) = \cos\tau$ are chosen, and besides ω_{k-1} and a_{k-1} $(k = 1, 2, 3, \ldots)$ are determined. So, the rule plays a very important role in the frame of the homotopy analysis method.

3. Some examples

In this section, the validity of the proposed approach is illustrated by two examples.

3.1. Example 1

First, consider the famous Van der Pol's equation [39,40]

$$\ddot{U}(t) + U(t) = \varepsilon[1 - U^2(t)]\dot{U}(t). \tag{39}$$

The corresponding perturbation approximation of the frequency

$$\omega \approx 1 - \frac{1}{16}\varepsilon^2 + \frac{17}{3072}\varepsilon^4 - \frac{678890}{5096079360}\varepsilon^6 + \cdots \tag{40}$$

is convergent in a quite small region $0 \leqslant \varepsilon < 2$.

Under transformation (3), Eq. (39) becomes

$$\omega^2 u''(\tau) + u(\tau) = \varepsilon\omega[1 - a^2u^2(\tau)]u'(\tau). \tag{41}$$

All other related equations are the same as defined in the Section 2, except the term R_k in Eq. (31) which becomes due to (33) that

$$R_k(\tau) = \sum_{n=0}^{k-1} u''_{k-1-n}(\tau)\left(\sum_{j=0}^{n}\omega_j\omega_{n-j}\right) + u_{k-1}(\tau) - \varepsilon\sum_{n=0}^{k-1}\omega_n u'_{k-1-n}(\tau) + \varepsilon\sum_{n=0}^{k-1}\left(\sum_{i=0}^{k-1-n}\omega_i u'_{k-1-n-i}(\tau)\right) \times\sum_{j=0}^{n}\left(\sum_{r=0}^{j}a_r a_{j-r}\right)\left(\sum_{s=0}^{n-j}u_s u_{n-j-s}\right). \tag{42}$$

Using symbolic calculation software Mathematica 4.1, it is easy to gain approximations at the first several orders. It is found that the frequency ω and the amplitude a at the mth-order of approximation can be expressed by

$$\omega \approx \omega_0 + \sum_{i=1}^{m}\varepsilon^{2i}\sum_{j=i}^{m}\alpha_m^{i,j}\hbar^j, \tag{43}$$

$$a \approx a_0 + \sum_{i=1}^{m-1}\varepsilon^{2i}\sum_{j=i+1}^{m}\beta_m^{i,j}\hbar^j, \tag{44}$$

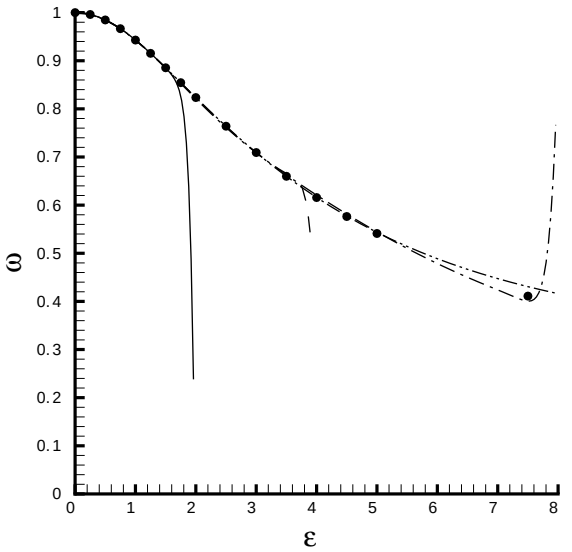

Fig. 1. Comparison of the frequency of the 20th-order homotopy analysis approximation with the exact result of Example 1. Symbols: exact solution; solid line: the perturbation result (40) and homotopy analysis approximation when $\hbar = -1$; dash line: $\hbar = -1/2$; dash-dot line: $\hbar = -1/5$; dash-dot-dot line: [10/10] Padé approximation of perturbation result (40).

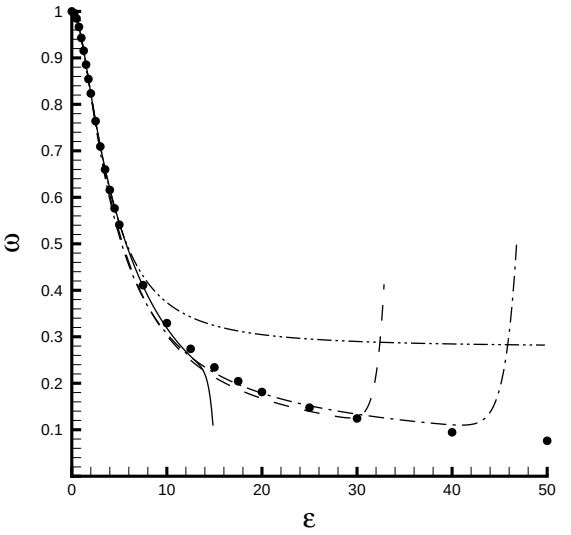

Fig. 2. Comparison of the frequency of the 20th-order homotopy analysis approximation with the exact result of Example 1. Symbols: exact solution; solid line: $\hbar = -1/\sqrt{1+\varepsilon^2}$; dash line: $\hbar = -1/\sqrt{1+4\varepsilon^2}$; dash-dot line: $\hbar = -1/\sqrt{1+8\varepsilon^2}$; dash-dot-dot line: [10/10] Padé approximation of perturbation result (40).

respectively, where the coefficients $\alpha_m^{i,j}, \beta_m^{i,j}$ depend upon m for given i, j, and $\omega_0 = 1$, $a_0 = 2$ are given by solving a set of non-linear algebraic equations

$$1 - \omega_0^2 = 0, \quad \varepsilon\omega_0 - \frac{\varepsilon}{4} a_0^2 \omega_0 = 0. \tag{45}$$

Note that results (43) and (44) contain the auxiliary parameter $\hbar$. It is found that convergence regions of the approximation series are dependent upon $\hbar$. The comparison of the frequency ω at the 20th-order of approximation with the numerical results is as shown in Fig. 1, where $\hbar = -1, -\frac{1}{2}$ and $-\frac{1}{5}$, respectively. Notice that $\hbar = -1$ gives exactly the same convergence region as the perturbation result (40). However, as $\hbar$ ($\hbar < 0$) closes to zero, the convergence region becomes larger and larger, although higher-order approximations are necessary to gain accurate enough results, as reported in our previous publications [23–28,13]. Therefore, $\hbar$ provides us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions when necessary.

Note that one has great freedom to choose the auxiliary parameter $\hbar$. Certainly, $\hbar$ can be chosen as a function of ε. Due to (43) and (44), the frequency ω and the amplitude a are even functions of ε. This implies that $\hbar$ should be an even function of ε, too. Consider three different functions of $\hbar$ as follows:

$$\hbar = -\frac{1}{\sqrt{1+\gamma\varepsilon^2}}, \tag{46}$$

$$\hbar = -\frac{1}{1+\gamma|\varepsilon|} \tag{47}$$

and

$$\hbar = -\exp\left[-\frac{\varepsilon^2}{1+\varepsilon^2/\gamma}\right], \tag{48}$$

respectively, where γ is a positive constant. As γ increases, the convergence regions of the frequency ω become larger and larger, as shown in Figs. 2–4. Notice that all of the 20th-order approximations agree with the exact result in larger convergence regions even than the [10/10] Padé approximation of the corresponding 20th-order perturbation result, as

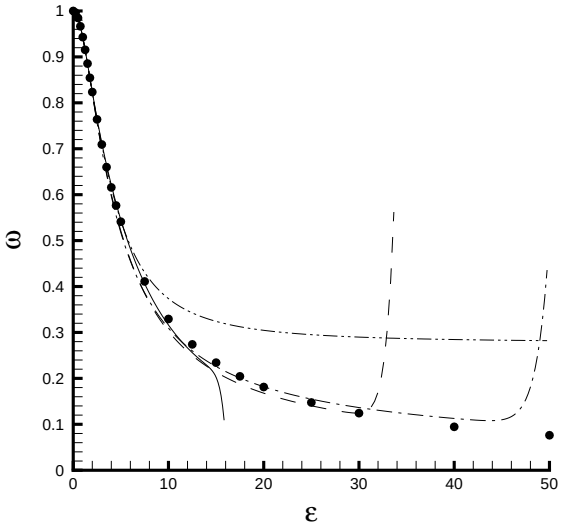

Fig. 3. Comparison of the frequency of the 20th-order homotopy analysis approximation with the exact result of Example 1. Symbols: exact solution; solid line: $\hbar=-(1+|\varepsilon|)^{-1}$; dash line: $\hbar=-(1+2|\varepsilon|)^{-1}$; dash-dot line: $\hbar=-(1+3|\varepsilon|)^{-1}$; dash-dot-dot line: [10/10] Padé approximation of perturbation result (40).

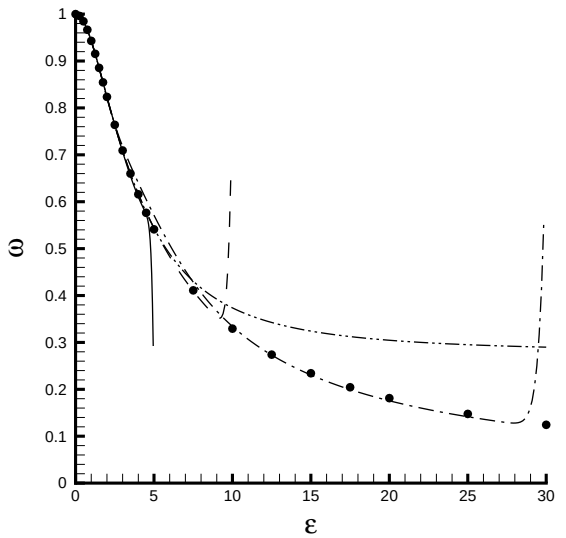

Fig. 4. Comparison of the frequency of the 20th-order homotopy analysis approximation with the exact result of Example 1. Symbols: exact solution; solid line: $\hbar=-\exp[-\varepsilon^2/(1+\varepsilon^2)]$; dash line: $\hbar=-\exp[-\varepsilon^2/(1+\varepsilon^2/2)]$; dash-dot line: $\hbar=-\exp[-\varepsilon^2/(1+\varepsilon^2/4)]$; dash-dot-dot line: [10/10] Padé approximation of perturbation result (40).

shown in Figs. 2–4. Note also that the perturbation approximation of the frequency is convergent only in a quite small region $0\leqslant\varepsilon<2$. Thus, the auxiliary parameter $\hbar$ indeed provides us with a convenient and flexible way to *control* the convergence of approximation series and *adjust* convergence regions, when necessary.

3.2. Example 2

Furthermore, consider (see [4, p. 135])

$$\ddot{U}(t)+U(t)-\varepsilon[1-U^2(t)]\dot{U}(t)+\varepsilon U^3(t)=0. \quad (49)$$

The corresponding perturbation approximation of the frequency

$$\omega\approx 1+\frac{3}{2}\varepsilon-\frac{7}{4}\varepsilon^2+\frac{143}{32}\varepsilon^3-\frac{11291}{768}\varepsilon^4+\frac{82919}{1536}\varepsilon^5-\frac{46617401}{221184}\varepsilon^6+\cdots \quad (50)$$

is convergent in a quite small region $0\leqslant\varepsilon<1/2$.

Under transformation (3), Eq. (49) becomes

$$\omega^2u''(\tau)+u(\tau)-\varepsilon\omega[1-a^2u^2(\tau)]u'(\tau)+\varepsilon a^2u^3(\tau)=0. \quad (51)$$

All other related equations are the same as defined in the Section 2, except the term R_k in Eq. (31) which becomes due to (33) that

$$R_k(\tau)=\sum_{n=0}^{k-1}u''_{k-1-n}(\tau)\left(\sum_{j=0}^{n}\omega_j\omega_{n-j}\right)+u_{k-1}(\tau)-\varepsilon\sum_{n=0}^{k-1}\omega_nu'_{k-1-n}(\tau)+\varepsilon\sum_{n=0}^{k-1}\left(\sum_{i=0}^{k-1-n}\omega_iu'_{k-1-n-i}(\tau)\right)\times\sum_{j=0}^{n}+\left(\sum_{r=0}^{j}a_ra_{j-r}\right)\left(\sum_{s=0}^{n-j}u_su_{n-j-s}\right)$$

$$+\varepsilon\sum_{n=0}^{k-1}u_{k-1-n}(\tau)\sum_{j=0}^{n}\left(\sum_{r=0}^{j}a_r a_{j-r}\right)\times\left(\sum_{s=0}^{n-j}u_s u_{n-j-s}\right). \tag{52}$$

Similarly, it is found that the frequency ω and the amplitude a at the mth-order of approximation can be expressed by

$$\omega\approx\omega_0^{1-2m}\left[\sum_{j=0}^{m}\gamma_m^{0,j}\varepsilon^j+\sum_{i=1}^{m}\left(\frac{\hbar^i}{\omega_0^{2i}}\right)\sum_{j=2}^{m+2i}\gamma_m^{i,j}\varepsilon^j\right], \tag{53}$$

$$a\approx a_0+\sum_{i=1}^{m}\left(\frac{\hbar^i}{\omega_0^{2i}}\right)\sum_{j=1}^{2m-1}\delta_m^{i,j}\varepsilon^j, \tag{54}$$

respectively, where the coefficients $\gamma_m^{i,j}$ and $\delta_m^{i,j}$ depend upon m for given i, j, and

$$\omega_0=\sqrt{1+3\varepsilon},\quad a_0=2 \tag{55}$$

are given by solving a set of non-linear algebraic equations

$$\varepsilon\omega_0-\frac{\varepsilon}{4}a_0^2\omega_0=0,\quad 1+\frac{3}{4}\varepsilon a_0^2-\omega_0^2=0. \tag{56}$$

Note also that approximation (53) and (54) contain the auxiliary parameter $\hbar$. It is found that in the case of constant $\hbar$ one gets the same qualitative conclusion as mentioned before, i.e. the closer $\hbar$ ($\hbar<0$) is to zero, the larger the convergence region of the series of the frequency, as shown in Fig. 5, where $\hbar=-1,-\frac{1}{2}$ and $-\frac{1}{4}$, respectively. Notice that when $\hbar=-\frac{1}{4}$ the frequency at the 10th-order of approximation can give good approximations in a region considerably larger than that of perturbation results, and is even better than the corresponding [5/5] Padé approximation.

Similarly, $\hbar$ can be a function of ε. It is found that when

$$\hbar=-\frac{1}{\sqrt{1+\varepsilon/2}} \tag{57}$$

the frequency at the 10th-order approximation agrees with numerical results in the region $0\leqslant\varepsilon\leqslant 500$,

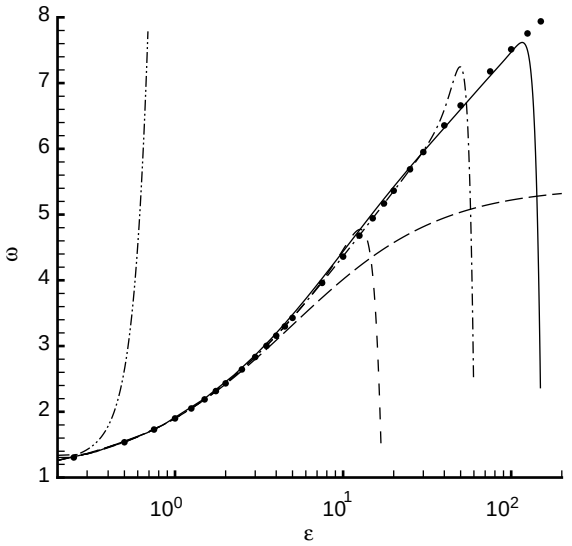

Fig. 5. Comparison of the frequency of the 10th-order homotopy analysis approximation with the exact frequency of Example 2. Symbols: exact solution; dash line: $\hbar=-1$; dash-dot line: $\hbar=-1/2$; solid line: $\hbar=-1/4$; dash-dot-dot line: perturbation result (50); long dash line: [5/5] Padé approximation of the perturbation expression (50).

which is nearly 1000 times larger than the perturbation result (50), as shown in Fig. 6. It is very interesting that, when setting $\hbar=-(1+\varepsilon/2)^{-1/2}$ in (53) and then expanding it in a power series of ε, one gains the same series as the perturbation power series (50). This is reasonable, because the power series of a given function is unique. Notice that the power series is only one kind of base functions. However, it is a fact that a given function can be represented *more efficiently* by a *better* set of base functions. This is the reason why when $\hbar=-(1+\varepsilon/2)^{-1/2}$ the approximation (53) converges in a region 1000 times larger than that of the perturbation power series (50) and is even much better than the corresponding [5/5] Padé approximations, as shown in Fig. 6.

All of these clearly illustrate that the auxiliary parameter $\hbar$ indeed provides us with a convenient and flexible way to *control* the convergence of approximation series and *adjust* convergence regions when necessary. Thus, the auxiliary parameter $\hbar$ plays an important role in the homotopy analysis method.

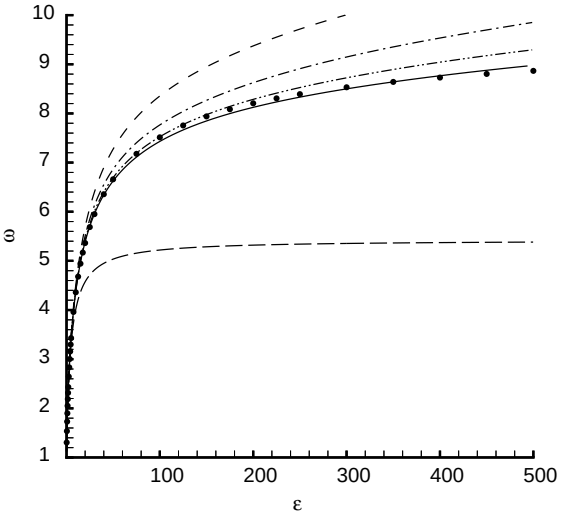

Fig. 6. Comparison of the frequency of the homotopy analysis approximation when $\hbar = -1/\sqrt{1+\varepsilon/2}$ with the exact frequency of Example 2. Symbols: exact solution; dash line: 4th-order approximation; dash-dot line: 6th-order approximation; dash-dot-dot line: 8th-order approximation; solid line: 10th-order approximation; long dash line: [5/5] Padé approximation of the perturbation expression (50).

4. Conclusion and discussion

In this paper the homotopy analysis method is employed to propose a new analytic approximate approach for limit-cycles of free oscillations of self-excited systems. Its validity is illustrated by two examples.

Notice that *the rule of solution expression* described by (8) plays an important role in the frame of the homotopy analysis method. It is under this rule that the auxiliary linear operator $\mathscr{L}$ and the initial guess $u_0(\tau) = \cos\tau$ are chosen, and besides ω_{k-1} and a_{k-1} $(k = 1, 2, 3, \ldots)$ are determined.

Different from all perturbation techniques on this topic, the proposed approach is valid even if a non-linear problem does *not* contain any small/large parameters. Thus, the proposed approach is rather general. Besides, unlike *all* other analytic techniques, the proposed approach provides us with a convenient way to *control* the convergence of approximation series and *adjust* convergence regions, *when necessary*. This is mainly because, unlike *all* other analytic techniques, the homotopy analysis method always gives a one-parameter *family* of analytic results in the auxiliary parameter $\hbar$, and besides one has great freedom to choose the value of $\hbar$. However, it is still not completely clear how to choose a proper (or better) value of $\hbar$ for *any* a given non-linear problem. Thus, further investigation and improvement are necessary.

Acknowledgements

This work is supported by National Science Fund for Distinguished Young Scholars of China (Approval No. 50125923).

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