A Second-Order Approximate Analytical Solution of a Simple Pendulum by the Process Analysis Method

S. J. Liao[1]

Institute of Shipbuilding,
University of Hamburg,
Laemmerseith 90,
D-2000 Hamburg 60,
Federal Republic of Germany

In this paper, a new kind of analytical method of nonlinear problem called the process analysis method (PAM) is described and used to give a second-order approximate solution of a simple pendulum. The PAM does not depend on the small parameter supposition and therefore can overcome the disadvantages and limitations of the perturbation expansion method. The analytical approximate results at the second-order of approximation are in good agreement with the numerical results. They are compared with perturbation solutions, and it appears that even the first-order solutions are more accurate than the perturbation solutions at second-order of approximation.

1 Introduction

It is difficult to solve nonlinear problems, either numerically or theoretically. Traditionally, iterative techniques were used to find numerical solutions of nonlinear problems, but nearly all iterative methods are sensitive to initial solutions. Thus, it is not easy to obtain converged results in cases of strong nonlinearity. On the other side, as mentioned by Nayfeh (1980) and O'Malley (1974), the perturbation expansion method is widely used to analyze simple nonlinear problems. It is well known that the perturbation method is based on *small* or *great* parameters. But, unfortunately, not every nonlinear problem has such *small* or *great* parameters. And it also seems difficult to decide whether or not a parameter is *small* or *great enough*.

For example, it is well known that the motion of a simple pendulum is periodical, which can be described mathematically as follows:

$$\begin{cases} \dfrac{d^2\theta}{dt^2}+\omega_0^2 \sin\theta=0 \\ \theta(t)=\beta \quad \text{for} \quad t=0 \\ \theta'(t)=0 \quad \text{for} \quad t=0 \end{cases} \tag{1}$$

where θ is the angle of swing, t is the time, β is the initial angle of swing, $\omega_0 = \sqrt{\dfrac{g}{l}}$; here, g is the gravity acceleration and l is the length of the simple pendulum.

It is easy to know that $|\theta(t)| \le \beta$. If the initial angle β is small enough, then θ is a small quantity and $\sin(\theta) \approx \theta$ is a good approximation; thus, the above equation has the solution

$$\theta(t)=\beta \cos(\omega_0 t) \tag{2}$$

with the period

$$T_0=\frac{2\pi}{\omega_0}. \tag{3}$$

But, if β is not small, the above solutions are not accurate. For example, when $\beta = 5\pi/9$, the numerical result of the period is $1.232T_0$. Therefore, higher-order approximate solutions should be given in this case. However, this is not easy, because Eq. (1) is nonlinear and has *no* small parameters. It seems doubtful to be able to give a good perturbation approximation of $\theta(t)$ and its corresponding period, especially in the case of the great initial angle β.

As mentioned by Ortega and Rheinboldt (1970), the continuous mapping technique, or so-called homotopy method, has been generally used to widen the domain of convergence of a given method or as a procedure to obtain sufficiently close starting points. The continuous mapping technique embeds a parameter that typically ranges from zero to one. When the embedding parameter is zero, the equation is one of the linear system. When it is one, the equation is the same as the original. Then, one can *iterate*, numerically along the solution path, by incrementing the imbedding parameter from zero to one; this continuously maps the initial linear solutions into the solutions of the original equation. Note that iterative techniques are used at each step along the solution path if the equation is nonlinear.

[1]Present address: Institute of Underwater Engineering, Department of Naval Architecture and Ocean Engineering, Shanghai Jiao Tong University, Shanghai 200030, P. R. China.

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for publication in the ASME Journal of Applied Mechanics.

Discussion on this paper should be addressed to the Technical Editor, Prof. Leon M. Keer, The Technological Institute, Northwestern University, Evanston, IL 60208, and will be accepted until four months after final publication of the paper itself in the Journal of Applied Mechanics.

Manuscript received by the ASME Applied Mechanics Division, May 5, 1990; final revision, Sept. 30, 1991. Associate Technical Editor: D. J. Inman.

If we derive the continuous mapping with respect to the imbedding parameter we will obtain the corresponding *linear* equations of this kind of derivatives. This is an interesting property of continuous mapping and a pure mathematical proof can be given. By means of this property of continuous mapping, a kind of general numerical method for nonlinear problems, called the finite process method (FPM), has been described by Liao (1991a, 1991b). The finite process method can avoid the use of iterative techniques to solve numerically nonlinear problems and it is insensitive to the initial solutions. Therefore, it can overcome the disadvantages and limitations of iterative techniques.

Based on the same property of continuous mapping, an analytical method for nonlinear problems, called the process analysis method (PAM), has been derived. It is interesting that this kind of analytical method *does not* depend on *small* or *great* parameters and therefore can overcome the limitations of perturbation techniques.

In this paper, the basic idea of the process analysis method is described and examined by using a simple pendulum as an example. The main purpose of this paper is to give a kind of general analytical method for nonlinear problems, which is *independent* on *small* or *great* parameters.

2 Basic Ideas of the Process Analysis Method

Let

$$\omega = \frac{2\pi}{T} \tag{4}$$

and

$$z = \omega t \tag{5}$$

where T is the period and ω is the frequence of a simple pendulum, respectively.

Then Eq. (1) is transformed into

$$\begin{cases} \dfrac{d^2\theta}{dz^2} + \lambda^2 \sin\theta = 0 \\ \theta(z) = \beta \quad \text{for } z = 0 \\ \theta'(z) = 0 \quad \text{for } z = 0. \end{cases} \tag{6}$$

Here,

$$\lambda = \frac{\omega_0}{\omega} = \frac{T}{T_0} \tag{7}$$

denotes the non-dimensional frequence or period of a simple pendulum.

A kind of continuous mapping, $\theta(z) \to \theta(z;p), \lambda \to \lambda(p)$ can be constructed as follows:

$$\begin{cases} \dfrac{\partial^2\theta(z;p)}{\partial z^2} + \lambda^2(p)\theta(z;p) \\ \quad + p\lambda^2(p)\{\sin[\theta(z;p)] - \theta(z;p)\} = 0 \\ \theta(z;p) = \beta \quad \text{for } z = 0 \\ \dfrac{\partial\theta(z;p)}{\partial z} = 0 \quad \text{for } z = 0 \end{cases} \tag{8}$$

where $p \in [0, 1]$, called the process-independent variable or imbedding variable.

For simplicity, call the continuous mapping $\theta(z;p)$ and $\lambda(p)$ *process*, or more precisely, *zero-order process*. Then, Eq. (8) could be called the *zero-order process equation*.

When $p = 0$, from zero-order process Eq. (8), one has the *initial equation*:

$$\begin{cases} \dfrac{\partial^2\theta(z;0)}{\partial z^2} + \lambda^2(0)\theta(z;0) = 0 \\ \theta(z;0) = \beta \quad \text{for } z = 0 \\ \dfrac{\partial\theta(z;0)}{\partial z} = 0 \quad \text{for } z = 0. \end{cases} \tag{9}$$

Denote $\lambda_0 = \lambda(0)$ and $\theta_0(z) = \theta(z;\theta)$. For simplicity, select $\lambda_0 = 1.0$, called the *initial solution* of $\lambda(p)$. It is easy to know that the above *linear* Eq. (9) has the solution

$$\theta_0(z) = \beta\cos(z). \tag{10}$$

When $p = 1.0$, from the zero-order process Eq. (8), one has the *final equation*

$$\begin{cases} \dfrac{\partial^2\theta(z;1.0)}{\partial z^2} + \lambda^2(1.0)\sin\theta(z;1.0) = 0 \\ \theta(z;1.0) = \beta \quad \text{for} \quad z = 0 \\ \dfrac{\partial\theta(z;1.0)}{\partial z} = 0 \quad \text{for} \quad z = 0. \end{cases} \tag{11}$$

Equations (11) are the same as (6). Denote that $\theta_f(z) = \theta(z;1.0)$ and $\lambda_f = \lambda(1.0)$, called the *final solution*. It is easy to understand that $\theta_f(z)$ and λ_f are just what we want to know.

From the above analysis, we can see that the zero-order process Eq. (8) gives a kind of relation between the initial solutions $\theta_0 = \beta \cos(z)$, $\lambda_0 = 1.0$ and the final solutions θ_f, λ_f. But this kind of relation is nonlinear, because the zero-order process Eq. (8) is generally a nonlinear one. In the following part of this section, a linear relation between θ_0, λ_0, and θ_f, λ_f will be introduced and used to give a kind of solution at the second-order approximation.

Define

$$\theta^{[k]}(z;p) = \frac{\partial^k\theta(z;p)}{\partial p^k} \tag{12}$$

$$\lambda^{[k]}(p) = \frac{\partial^k\lambda(p)}{\partial p^k} \tag{13}$$

as the kth-order *process derivative* of $\theta(z;p)$ and $\lambda(p)$, respectively.

Suppose:

1 $\theta(z;p)$, $\lambda(p)$ have definition in $p \in [0, 1]$, $0 \le z < \infty$ and
2 there exist $\theta^{[k]}(z;p)$ and $\lambda^{[k]}(p)$ in $p \in [0, 1]$, $0 \le z < \infty$ for $k \ge 1$

then, according to Taylor's theory, $\theta_0(z)$, λ_0 and $\theta_f(z)$, λ_f have the following relations:

$$\theta_f(z) = \theta_0(z) + \sum_{k=1}^{\infty} \frac{\theta^{[k]}(z;p)}{k!}\bigg|_{p=0} \tag{14}$$

$$\lambda_f = \lambda_0 + \sum_{k=1}^{\infty} \frac{\lambda^{[k]}(p)}{k!}\bigg|_{p=0} \tag{15}$$

where $k! = 1 \times 2 \times \cdots \times (k-1) \times k$ is the factorial of k. $\theta^{[k]}(z;p), \lambda^{[k]}(p)$ at $p = 0$ can be obtained in the following way.

Deriving the zero-order process Eq. (8) with respect to p, one can obtain the *first-order process equation* as follows:

$$\begin{cases} \dfrac{\partial^2\theta^{[1]}(z;p)}{\partial z^2}+\lambda^2(p)\theta^{[1]}(z;p)+2\lambda(p)\lambda^{[1]}(p)\theta(z;p) \\ +\lambda^2(p)\{\sin[\theta(z;p)]-\theta(z;p)\} \\ +2p\lambda(p)\lambda^{[1]}(p)\{\sin[\theta(z;p)]-\theta(z;p)\} \\ +p\lambda^2(p)\{\cos\theta(z;p)-1\}\theta^{[1]}(z;p)=0 \\ \theta^{[1]}(z;p)=0 \quad \text{for } z=0 \\ \dfrac{\partial\theta^{[1]}(z;p)}{\partial z}=0 \quad \text{for } z=0. \end{cases} \tag{16}$$

When $p = 0.0$, from first-order process Eq. (16), one has

$$\begin{cases} \dfrac{\partial^2\theta^{[1]}(z;0)}{\partial z^2}+\lambda_0^2\theta^{[1]}(z;0) \\ =\lambda_0^2\{\theta_0(z)-\sin\theta_0\}-2\lambda_0\lambda^{[1]}(0)\theta_0 \\ \theta^{[1]}(z;0)=0 \quad \text{for } z=0 \\ \dfrac{\partial\theta^{[1]}(z;0)}{\partial z}=0 \quad \text{for } z=0. \end{cases} \tag{17}$$

It can be derived that

$$\sin(\theta_0)=2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(\beta)\cos(2m+1)z \tag{18}$$

and

$$\cos(\theta_0)=J_0(\beta)+2\sum_{m=1}^{\infty}(-1)^m J_{2m}(\beta)\cos(2mz) \tag{19}$$

where, $J_n(\beta)$ is first-sort of Bessel function denoted as

$$J_n(\beta)=\sum_{k=0}^{\infty}\frac{(-1)^k(\beta/2)^{2k+n}}{k!(k+n)!}. \tag{20}$$

Substituting (18) in (17), one has the first-order process equation at $p = 0$ as follows:

$$\begin{cases} \dfrac{\partial^2\theta^{[1]}(z;0)}{\partial z^2}+\theta^{[1]}(z;0)=\{\beta-2J_1(\beta)-2\beta\lambda_0^{[1]}\}\cos(z) \\ \quad -2\displaystyle\sum_{m=1}^{\infty}(-1)^m J_{2m+1}(\beta)\cos(2m+1)z \\ \theta^{[1]}(z;0)=0, \quad \text{for } z=0 \\ \dfrac{\partial\theta^{[1]}(z;0)}{\partial z}=0, \quad \text{for } z=0. \end{cases} \tag{21}$$

In order to let the above Eq. (21) have finite solution (i.e., the secular terms should be eliminated), it must be satisfied that

$$\lambda^{[1]}(0)=\frac{1}{2}-\frac{J_1(\beta)}{\beta}$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\beta}{2}\right)^{2k}\frac{1}{k!(k+1)!}$$

$$=\frac{\beta^2}{16}-\frac{1}{24}\left(\frac{\beta}{2}\right)^4+\cdots. \tag{22}$$

Thus, the *linear* differential Eq. (21) has solution

$$\theta^{[1]}(z;0)=\sum_{m=0}^{\infty}a_m\cos(2m+1)z, \tag{23}$$

where

$$a_0=-\sum_{m=1}^{\infty}a_m \tag{24}$$

$$a_m=\frac{(-1)^m J_{2m+1}(\beta)}{2m(m+1)} \quad (m\geq 1). \tag{25}$$

Deriving the first-order process Eq. (16) with respect to p and then let $p = 0$, one can obtain the *second-order process equation* at $p = 0$ as follows:

$$\begin{cases} \dfrac{\partial^2\theta^{[2]}(z;0)}{\partial z^2}+\theta^{[2]}(z;0) \\ =4\lambda^{[1]}(0)\{\theta_0-\sin\theta_0\}+2\{1-\cos\theta_0)\theta^{[1]}(z;0) \\ \quad -4\lambda^{[1]}(0)\theta^{[1]}(z;0)-2\{(\lambda^{[1]})^2+\lambda^{[2]}(0)\}\theta_0 \\ \theta^{[2]}(z;0)=0 \quad \text{for } z=0 \\ \dfrac{\partial\theta^{[2]}(z;0)}{\partial z}=0 \quad \text{for } z=0. \end{cases} \tag{26}$$

The above equation is linear with respect to $\theta^{[2]}(z;0)$. Substituting (18), (19), (22), (23) in (26) and eliminating the secular terms, we have

$$\lambda_0^{[2]}=3\left[\frac{1}{2}-\frac{J_1(\beta)}{\beta}\right]^2-\frac{J_2(\beta)J_3(\beta)}{4\beta}$$
$$-\frac{1}{\beta}\left[\frac{J_1(\beta)}{\beta}+\frac{J_2(\beta)-J_0(\beta)}{2}\right]\sum_{m=1}^{\infty}\frac{(-1)^m J_{2m+1}(\beta)}{m(m+1)}$$
$$-\sum_{m=2}^{\infty}\frac{J_{2m}(\beta)}{2m\beta}\left[\frac{J_{2m+1}(\beta)}{m-1}-\frac{J_{2m-1}(\beta)}{m-1}\right] \tag{27}$$

and

$$\theta^{[2]}(z;0)=\sum_{m=0}^{\infty}b_m\cos(2m+1)z$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{mn}\cos[2(n+m)+1]z$$
$$+\sum_{m=1}^{\infty}\left(\sum_{\substack{n=1\\ n\neq m\\ n\neq m-1}}^{\infty}\right)d_{mn}\cos[2(n-m)+1]z \tag{28}$$

where

$$b_0=-\sum_{m=1}^{\infty}b_m-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{mn}-\sum_{m=1}^{\infty}\left(\sum_{\substack{n=1\\ n\neq m\\ n\neq m-1}}^{\infty}\right)d_{mn} \tag{29}$$

$$b_m=\frac{(-1)^m J_{2m+1}(\beta)}{2m(m+1)}\left\{2-\frac{4J_1(\beta)}{\beta}-\frac{1}{m(m+1)}\left[\frac{J_1(\beta)}{\beta}-\frac{J_0(\beta)}{2}\right]\right\}$$
$$+\frac{(-1)^m a_0[J_{2m}(\beta)-J_{2m+2}(\beta)]}{2m(m+1)} \quad (m\geq 1) \tag{30}$$

$$c_{mn}=\frac{(-1)^{m+n}J_{2m}(\beta)J_{2n+1}(\beta)}{4n(n+1)(n+m)(n+m+1)} \quad (m\geq 1, n\geq 1) \tag{31}$$

$$d_{mn}=\frac{(-1)^{m+n}J_{2m}(\beta)J_{2n+1}(\beta)}{4n(n+1)(n-m)(n-m+1)}$$
$$(m\geq 1; n\geq 1,\ n\neq m,\ n\neq m-1). \tag{32}$$

2.1 Approximation of Frequence ω.

According to

$$\omega(p)=\frac{\omega_0}{\lambda(p)} \tag{33}$$

and Taylor's theory, one has the first-order approximation of frequence ω as

$$\omega_1=\omega_0\left\{\frac{1}{\lambda(p)}-\frac{\lambda^{[1]}(p)}{\lambda^2(p)}\right\}\Bigg|_{p=0}$$

$$=\omega_0(1-\lambda_0^{[1]}) \tag{34}$$

and second-order approximation of frequence as

$$\omega_2=\omega_0\left\{\frac{1}{\lambda(p)}-\frac{\lambda^{[1]}(p)}{\lambda^2(p)}+\frac{[\lambda^{[1]}(p)]^2}{\lambda^3(p)}-\frac{\lambda^{[2]}(p)}{2\lambda^2(p)}\right\}\Bigg|_{p=0}$$

$$=\omega_0\left\{1-\lambda_0^{[1]}-\frac{1}{2}\lambda_0^{[2]}+(\lambda_0^{[1]})^2\right\}. \tag{35}$$

2.2 Approximation of Period *T*. According to $T/T_0=\omega_0/\omega$, one has the first-order approximation of period T as

$$\frac{T_1}{T_0}=\frac{1}{1-\lambda_0^{[1]}}$$

$$=1+\frac{\beta^2}{16}+\frac{1}{48}\left(\frac{\beta}{2}\right)^4+\cdots \tag{36}$$

and the second-order approximation of period as

$$\frac{T_2}{T_0}=\frac{1}{1-\lambda_0^{[1]}-\frac{1}{2}\lambda_0^{[2]}+(\lambda_0^{[1]})^2}. \tag{37}$$

2.3 Approximation of $\theta(t)$. $\theta(t)$ at first-order of approximation is

$$\theta_1(t)=(\beta+a_0)\cos z+\sum_{m=1}^{\infty}a_m\cos(2m+1)z$$

$$=(\beta+a_0)\cos z-\frac{J_3(\beta)}{4}\cos(3z)+\frac{J_5(\beta)}{12}\cos(5z)$$

$$-\frac{J_7(\beta)}{24}\cos(7z)+\cdots=\left\{\beta+\frac{\beta^3}{192}-\frac{1}{90}\left(\frac{\beta}{2}\right)^5+\cdots\right\}\cos(\omega_1 t)$$

$$-\left\{\frac{\beta^3}{192}-\frac{1}{96}\left(\frac{\beta}{2}\right)^5+\cdots\right\}\cos(3\omega_1 t)$$

$$+\left\{\frac{1}{1440}\left(\frac{\beta}{2}\right)^5+\cdots\right\}\cos(5\omega_1 t)+\cdots \tag{38}$$

and $\theta(t)$ at second-order of approximation is

$$\theta_2(t)=\left(\beta+a_0+\frac{b_0}{2}\right)\cos(\omega_2 t)+\sum_{m=1}^{\infty}\left(a_m+\frac{b_m}{2}\right)$$

$$\times\cos[(2m+1)\omega_2 t]+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{c_{mn}}{2}\cos[2(n+m)+1](\omega_2 t)$$

$$+\sum_{m=1}^{\infty}\left(\sum_{\substack{n=1\\ n\neq m\\ n\neq m-1}}^{\infty}\right)\frac{d_{mn}}{2}\cos[2(n-m)+1](\omega_2 t). \tag{39}$$

As the last part of this section, let us discuss simply the expressions (14), and (15). The expressions (14), (15) describe a kind of relation between the initial solution $\theta_0=\beta\cos(z)$, $\lambda_0=1.0$ and the final solution θ_f, λ_f by means of kth-order process derivatives $\theta^{[k]}(z;p)$ and $\lambda^{[k]}(p)$ at $p=0$. The first-order process Eq. (17) and the second-order process Eq. (26) are *linear* with respect to $\theta^{[1]}(z;0)$ and $\theta^{[2]}(z;0)$, respectively. One can prove easily that the kth-order process equation ($k=1, 2, 3, \cdots$) is *alway* linear with respect to $\theta^{[k]}(z;0)$. Therefore, $\theta^{[1]}(z;0)$, $\theta^{[2]}(z;0), \cdots, \theta^{[k]}(z;0), \cdots$, can be obtained without great difficulties. It means that the expressions (14), (15) give a kind of linear relation between the initial solution and final solution, although the zero-order process Eq. (8) is generally nonlinear. Using process derivatives, a nonlinear problem with respect to $\theta(z)$ can be transformed into an infinite number of linear problems with respect to $\theta^{[k]}(z;p)$ ($k=1, 2, 3, \cdots, \infty$) at $p=0$. But, only a finite number of linear problems with respect to $\theta^{[k]}(z;p)$($k=1, 2, \cdots, n_p$) at $p=0$ can be solved. It means that a nonlinear problem can be approximated by a finite number of linear problems. One would say that a nonlinear problem could be discretized into n_p linear problems with respect to kth-order process derivatives ($k=1, 2, \cdots, n_p$). The greater n_p is, the more exact this approximation is. These are the basic ideas of the Process Analysis Method.

3 Comparisons to the Numerical and Perturbation Results

In order to examine the solutions given by PAM, it is valuable to compare them to the numerical results and perturbation solutions at the same order of approximation.

The original Eq. (1) can be solved numerically by means of Runge-Kutta's method. In numerical computation we select $\Delta t=0.0001$ second and use double precision variables in the computer program.

On the other side, substitute

$$\sin(\theta)=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\cdots \tag{40}$$

in (6) and *suppose* that β be *small enough* and θ and λ could be described, respectively, as

$$\theta=\beta\{\Theta^{(0)}+\beta\Theta^{(1)}+\beta^2\Theta^{(2)}+\cdots\}, \tag{41}$$

$$\lambda=1+\beta\Lambda^{(1)}+\beta^2\Lambda^{(2)}+\cdots, \tag{42}$$

then, we can obtain

(a) perturbation solutions at first-order of approximation:

$$\theta(t)=\beta\cos(\omega_0 t) \tag{43}$$

$$\frac{T_1}{T_0}=1, \tag{44}$$

(b) perturbation solutions at second-order of approximation:

$$\theta(t)=\left(\beta+\frac{\beta^3}{192}\right)\cos\left(\frac{\omega_0 t}{1+\frac{\beta^2}{16}}\right)-\frac{\beta^3}{193}\cos\left(\frac{3\omega_0 t}{1+\frac{\beta^2}{16}}\right) \tag{25}$$

$$\frac{T_2}{T_0}=1+\frac{\beta^2}{16}. \tag{46}$$

3.1 Comparison of Nondimensional Period T/T_0. The numerical and analytical results show that the nondimensional period T/T_0 of a simple pendulum is only dependent on the initial angle β. The detailed comparison of numerical and analytical results of T/T_0 obtained, respectively, by PAM and the perturbation method is given in Table 1. It seems that PAM solutions, even at first-order of approximation, generally agree

Table 1 Comparison of theoretical and numerical results of T/T_0

β	numerical method	PAM		perturbation method (second-order)
		first-order	second-order	
20°	1.008	1.008	1.008	1.008
30°	1.017	1.017	1.017	1.017
40°	1.031	1.031	1.031	1.030
50°	1.050	1.048	1.050	1.048
60°	1.073	1.070	1.073	1.069
70°	1.102	1.096	1.101	1.093
80°	1.138	1.127	1.136	1.122
90°	1.180	1.162	1.177	1.154
100°	1.232	1.201	1.225	1.190
110°	1.295	1.246	1.282	1.230
120°	1.373	1.296	1.347	1.274
130°	1.471	1.351	1.424	1.322

better with the numerical results than perturbation solutions at second-order of approximation. Even in the case of a great initial angle $\beta = 130$ deg, the second-order PAM solution can also give a good enough approximate value of a period, but the perturbation solution at the same-order of approximation has about a ten percent relative error.

3.2 Spectrum Analysis. It is well known that the solution $\theta(t)$ of the original Eq. (1) is a periodical function. So, $\theta(t)$ can be expressed in the form of a Fourier series.

Table 2 Spectrum analysis: Numerical result

β	30°	60°	90°	120°
A_1/β	1.001	1.006	1.015	1.030
A_2/β	3.29E-6	-2.21E-7	4.77E-8	3.38E-6
A_3/β	-1.45E-3	-6.14E-3	-1.53E-2	-3.20E-2
A_4/β	5.90E-7	-8.39E-8	-4.24E-8	1.43E-7
A_5/β	3.43E-6	6.62E-5	3.96E-4	1.65E-3
A_6/β	2.25E-7	-6.85E-8	-4.86E-8	1.41E-7
A_7/β	-1.61E-7	-7.84E-7	-1.22E-5	-1.01E-4

Table 3 Spectrum analysis: First-order PAM solutions

β	30°	60°	90°	120°
A_1/β	1.001	1.005	1.011	1.017
A_3/β	-1.40E-3	-5.33E-3	-1.10E-2	-1.72E-2
A_5/β	1.61E-6	2.49E-5	1.19E-4	3.47E-4
A_7/β	-1.32E-9	-8.23E-8	-8.95E-7	-4.70E-6

Table 4 Spectrum analysis: Second-order PAM solutions

β	30°	60°	90°	120°
A_1/β	1.001	1.006	1.014	1.025
A_3/β	-1.45E-3	-6.03E-3	-1.41E-2	-2.60E-2
A_5/β	1.96E-6	4.57E-5	3.29E-4	1.34E-3
A_7/β	-1.88E-9	-2.22E-7	-4.27E-6	-3.57E-5

According to (38) and (39), the PAM solutions, at both first and second-order of approximation, can be written in the form of

$$\theta(t) = \sum_{k=1}^{\infty} A_{2k-1} \cos\,[(2k-1)\omega t] \qquad (k=1,\,2,\cdots). \quad (47)$$

Numerically, we have

$$\overline{A}_n = \frac{4}{T}\int_0^{T/2} \theta(t)\,\cos(n\omega t)dt \qquad (n=1,\,2,\cdots) \quad (48)$$

where the numerical solution of $\theta(t)$ is used, and thus a numerical integral over $[0, T/2]$ is needed.

The numerical values $\overline{A}_k (k = 1, 2, \cdots)$ and their corresponding analytical values at first and second-order of approximation given by PAM are shown as Table 2, Table 3, and Table 4, respectively. These results show that the second-order PAM solutions are in better agreement with the numerical solutions than the first-order PAM results. Note that the analytical values of $A_{2k} (k = 1, 2, \cdots)$ are zero and the corresponding numerical values of $\overline{A}_2$, $\overline{A}_4$, and $\overline{A}_6$ given in Table 2 are so small that they can be as the numerical errors in the integral. It appears that PAM captures the leading nonlinear harmonics found in the numerical solutions.

The numerical results and analytical solutions at second-order of approximation given, respectively, by the perturbation method and PAM, in case of $\beta = 5\pi/9$, are shown Fig. 1.

The analytical and numerical values of $A_k (k > 1)$ are so much smaller than A_1 that

$$\theta(t) = \beta\,\cos(\omega_2 t) \quad (49)$$

can give a good enough approximation of the original Eq. (1). This simplified expression gives essentially the same results as the second-order PAM solutions (39).

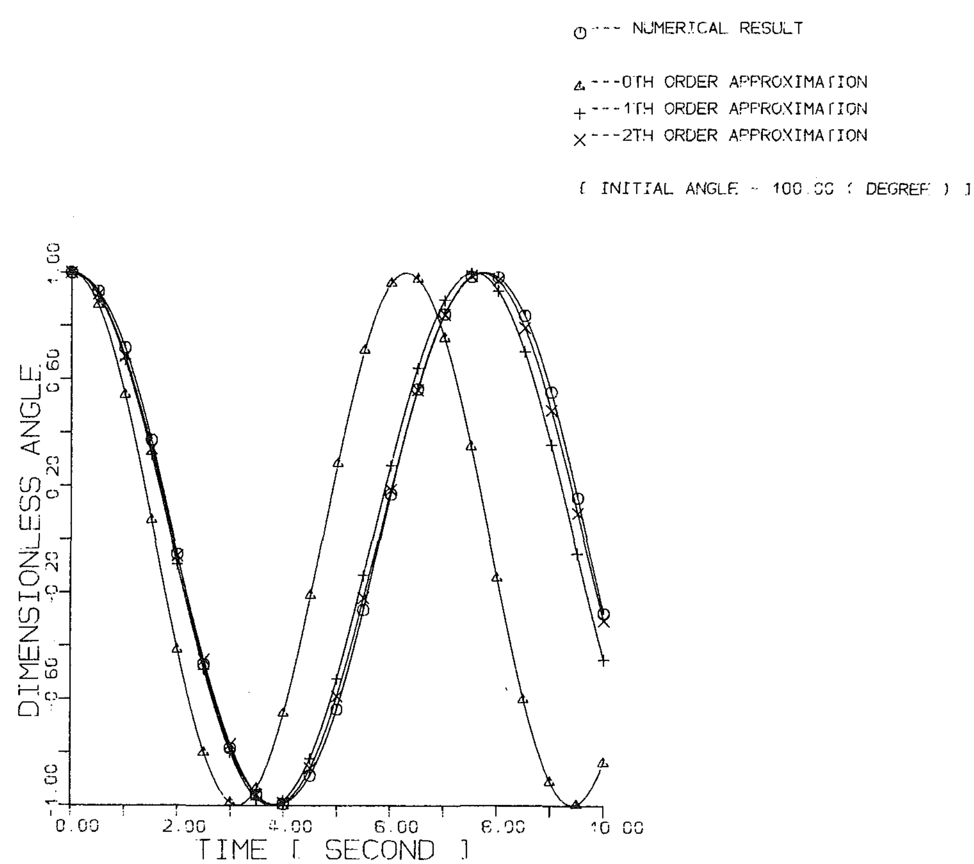

Fig. 1 Comparison of the analytical solutions at second-order of approximation to the numerical results

With comparison of expressions (36), (38), to expressions (46) and (45), it appears that the first-order PAM solutions include the terms of the perturbation solutions at second-order of approximation.

4 Conclusion and Discussion

In this paper, the basic idea of a new kind of analytical method for nonlinear problems, called PAM, is described and used to give a second-order approximate solutions of a simple pendulum. These solutions are compared to the numerical and perturbation solutions. The comparison shows that even the first-order PAM solutions agree better with the numerical solutions than the second-order perturbation solutions. So, we have reason to believe that PAM can indeed give more accurate analytical results than the perturbation method. Note that small or great parameter supposition is *not* needed for PAM. This is an advantage of PAM.

The perturbation method seems like a kind of art. Especially in the case of singular perturbation problems one must use different techniques, for example, the methods of multiple-scale expansions, the method of matched asymptotic expansions and so on, to solve different problems. Therefore, experiences seem important for the perturbation method. But, contrary to perturbation techniques, PAM has the simplicity in logic. This is another advantage of PAM.

The process analysis method is based on the two concepts of *process* and *process derivatives*. Process is a kind of continuous mapping which connects the initial solutions with the solutions of the original nonlinear problem. But this continuous mapping described by the zero-order process equation is generally nonlinear for a nonlinear problem. It is interesting and important that the kth-order process equations are linear with respect to the kth-order process derivatives. Then, according to Taylor's theory, the final solution and initial solution can be connected by the kth-order process derivatives ($k = 1, 2, \cdots$). That is why the process derivatives should be used.

The process analysis method is also based on suppositions, i.e., there should exist zero-order process and kth-order process derivatives, and the corresponding Taylor's series should converge to the solution of the original equations. Therefore, PAM is also dependent on suppositions, but these suppositions are *not* small-parameter suppositions. Owing to this reason, one can use it to solve more nonlinear problems, especially those without small or great parameters.

However, it should be pointed out that Taylor's series has generally a *finite* radius of convergence. (Only few functions have converged Taylor's series with infinite radius of convergence; for example, $\sin(x)$, $\cos(x)$, and so on.) PAM requires the kth-order process derivatives ($k = 1, 2, \cdots$) to exist, and to be well behaved and be bounded (i.e., they must not have turning points or diverge to infinity in $p \in [0, 1]$), and converge. Up to now, it is unknown whether or not the corresponding Taylor's series, given by PAM, have *always* a larger radius of convergence than an asymptotic series based on the perturbation method. These are the limitations of PAM.

Acknowledgments

Professors W. H. Isay and H. Soeding are acknowledged for their suggestions. Thanks to the DAAD for the financial support.

References

Liao, S. J., 1991a, "A General Numerical Method for The Solution of Gravity Wave Problems. Part 1: 2D Steep Gravity Waves in Shallow Water," *International Journal for Numerical Methods in Fluids*, Vol. 12, pp. 727–745.

Liao, S. J., 1991b, "The numerical limiting form of 2D nonlinear gravity waves past a submerged vortex by Finite Process Method (FPM)," *The Sixth International Workshop on Water Waves and Floating Bodies*, Woods Hole, Mass., pp. 139–143.

Nayfeh, A. H., 1980, *International to Perturbation Techniques*, John Wiley and Sons, New York.

Ortega, J. M., and Rheinboldt, W. C., 1970, *Iterative Solution of Nonlinear Equations in Several Variables*, New York, Academic Press, pp. 230–235.

O'Malley, R. E., 1974, *Introduction to Singular Perturbations*, Academic, New York.