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MATHEMATICAL PROOF OF SPATIAL SYMMETRY INVARIANCE OF TWO-DIMENSIONAL KOLMOGOROV FLOW

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We prove a mathematical theorem that the exact solution of the two-dimensional (2D) Kolmogorov flow governed by the Navier–Stokes (NS) equations with the periodic boundary condition maintains the same spatial symmetry as its initial condition globally in time, i.e., for all $t \geqslant 0$. This mathematical theorem can be used as an absolute scale to check the correctness and reliability of numerical simulations of NS turbulence. It is found that all reported CNS (clean numerical simulation) results of the NS turbulence maintain the same spatial symmetry as the corresponding initial condition in the whole time interval of simulation, but all of the corresponding DNS (direct numerical simulation) results lose their spatial symmetry quickly. In other words, these DNS results violate this mathematical theorem. Therefore, this mathematical theorem rigorously confirms that the spatiotemporal trajectories of NS turbulence given by DNS are indeed quickly polluted by numerical noises badly. It also illustrates that CNS can provide helpful enlightenment to deepen our understanding of turbulence, since it can reveal mathematical truths and realities of NS turbulence.

Keywords: Navier–Stokes turbulence, Kolmogorov flow, spatial symmetry, CNS, DNS

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1. Introduction

Lorenz [1], who put forward in 1963 the famous "butterfly-effect" of chaos [2]–[5], further discovered in 2006 that numerical errors could have significant influences on the global properties of chaotic systems [6], because unavoidable numerical noises as a small artificial disturbance exponentially increase to a macro-level due to the butterfly-effect. Note that the Navier–Stokes (NS) turbulence [7]–[10] (i.e., turbulence governed by the NS equations) is chaotic [11]–[15]. However, for all traditional numerical algorithms even including the direct numerical simulation (DNS) [16]–[20] of the NS equations, numerical noises are unavoidable. Thus, logically speaking, according to the butterfly-effect of a chaotic system, the numerical noises of the DNS of NS turbulence should quickly increase to a macro-level, which certainly should lead to a distinct departure of the numerical trajectory from the exact solution of the NS equations. Thus, logically speaking, DNS of NS turbulence should be quickly polluted by numerical noises badly.

However, DNS results have been widely regarded as the most accurate and thus have been used as benchmark solutions of turbulent flows for quite a long time [16]–[20]. Thus, it is impossible to use the traditional algorithms to investigate the accuracy and validity of DNS. In 2009, Liao [21] proposed the so-called "clean numerical simulation" (CNS) [22]–[33] to overcome the restrictions of the traditional numerical

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algorithms for chaos and turbulence. Unlike DNS, numerical noises of CNS can be rigorously negligible in a finite time-interval, which is however long enough for calculating reliable statistics, by means of reducing both the truncation error and the round-off error to a required small enough level [21]–[33]. Note that the number of discovered periodic orbits of the famous planar three-body problems has been increased by several orders of magnitude by means of the CNS [24]–[26]: this is a good example to illustrate the potential and power of CNS. Thus, CNS can give very accurate spatiotemporal trajectories of the NS equations in a finite but long enough interval of time: they are very close to the exact solution, i.e., the truth and reality of NS turbulence.

Note that the numerical noises of CNS are much smaller than those of DNS. Thus, the CNS results can be used as benchmark solutions to check the correctness and reliability of DNS. In fact, some comparisons of CNS and DNS were given for the two-dimensional (2D) incompressible turbulent Kolmogorov flow [31], [32]: the numerical simulations given by CNS maintain the same spatial symmetry in the whole time-interval of simulation as the corresponding initial condition, but the corresponding DNS results maintain the same spatial symmetry only at the beginning but quickly lose it. *Which one reveals the truth and reality of the NS turbulence*?

In this paper, we prove a mathematical theorem that the exact solution of the 2D Kolmogorov flow considered in [31], [32] must always maintain the same spatial symmetry as its initial condition forever, i.e., for all $t \geqslant 0$. Note that the CNS results [31], [32] agree well with this mathematical theorem and thus reveal the truth and reality, but the corresponding DNS results quickly lose the spatial symmetry and thus *violate* this mathematical theorem, clearly confirming that DNS results are indeed quickly polluted by numerical noises badly and depart distinctly away from the exact solution. The detailed mathematical proofs are given in Sec. 2, and the concluding remarks and discussions are given in Sec. 3.

2. Spatial symmetry invariance of Kolmogorov flow

Consider a 2D incompressible Kolmogorov flow [34]–[36] in a square domain under the so-called Kolmogorov forcing, governed by the dimensionless Navier–Stokes equation in the form of the stream-function

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x - \frac{1}{\mathrm{Re}} \nabla^4 \psi + n_{\mathrm{K}} \cos(n_{\mathrm{K}} y) = 0, \tag{1}$$

where ψ is the stream-function, t denotes the time, $x, y \in [0, 2\pi]$ are horizontal and vertical coordinates in the coordinate system xOy, Re is the Reynolds number, n_{K} is an even number describing the Kolmogorov forcing scale, ∇^2 is the Laplace operator, and $\nabla^4 = \nabla^2\nabla^2$, respectively. The stream-function ψ satisfies the periodic boundary condition

$$\psi(x, y, t) = \psi(x \pm 2\pi, y, t) = \psi(x, y \pm 2\pi, t). \tag{2}$$

Here, let us consider a smooth initial condition $\psi(x, y, 0)$ that contains the spatial symmetry of rotation:

$$\psi(x, y, 0) = \psi(2\pi - x, 2\pi - y, 0) \tag{3}$$

and/or translation:

$$\psi(x, y, 0) = \psi(x + \pi, y + \pi, 0). \tag{4}$$

Define

$$\psi^{(m)}(x,y,t) = \frac{1}{m!}\frac{\partial^m \psi(x,y,t)}{\partial t^m}, \qquad \psi^{(0)}(x,y,t) = \psi(x,y,t). \tag{5}$$

Assume that $\psi(x,y,t)$ can be expanded as a Taylor series at $t = t_0 \in [0,+\infty)$ with a nonzero radius $\rho > 0$ of convergence, written as

$$\psi(x,y,t) = \psi(x,y,t_0) + \sum_{m=1}^{+\infty} \psi^{(m)}(x,y,t_0)(t-t_0)^m = \sum_{m=0}^{+\infty} \psi^{(m)}(x,y,t_0)(t-t_0)^m, \tag{6}$$

where $|t-t_0| < \rho$. Then, we can rewrite Eq. (1) as follows:

$$\nabla^2\psi^{(1)}(x,y,t) + \psi_x^{(0)}\nabla^2\psi_y^{(0)} - \psi_y^{(0)}\nabla^2\psi_x^{(0)} - \frac{1}{\mathrm{Re}}\nabla^4\psi^{(0)} + n_{\mathrm{K}}\cos(n_{\mathrm{K}}y) = 0. \tag{7}$$

Let us consider another coordinate system $O' - x'y'$, where

$$x' = 2\pi - x, \qquad y' = 2\pi - y \tag{8}$$

for rotation, and

$$x' = \pi + x, \qquad y' = \pi + y \tag{9}$$

for translation, respectively. Let $\psi'(x',y',t)$ be the solution of Eq. (1) in the new coordinates:

$$\nabla'^{\,2}\psi'^{(1)} + \psi'_{x'}\nabla'^{\,2}\psi'_{y'} - \psi'_{y'}\nabla'^{\,2}\psi'_{x'} - \frac{1}{\mathrm{Re}}\nabla'^{\,4}\psi' + n_{\mathrm{K}}\cos(n_{\mathrm{K}}y') = 0 \tag{10}$$

under the periodic boundary condition, subject to an initial condition $\psi'(x',y',0)$. If there exists a spatial symmetry of rotation (3) and/or translation (4) for the initial solution, it holds $\psi'(x',y',0) = \psi(x,y,0)$.

Lemma 1 (operator covariance). *Let $n_{\mathrm{K}} \geqslant 2$ denote an even integer, and let f and g denote two smooth functions with the spatial symmetry $f(x,y,t) = f'(x',y',t)$ and $g(x,y,t) = g'(x',y',t)$, respectively. It holds*

$$\cos(n_{\mathrm{K}}y) = \cos(n_{\mathrm{K}}y'), \qquad \nabla^2 = \nabla'^{\,2}, \qquad \nabla^4 = \nabla'^{\,4}, \qquad \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} = \frac{\partial f'}{\partial x'}\frac{\partial g'}{\partial y'} \tag{11}$$

in the case of (8) *and/or* (9).

Proof. In the case of (8), we have

$$\begin{aligned}
&\cos(n_{\mathrm{K}}y') = \cos(2n_{\mathrm{K}}\pi - n_{\mathrm{K}}y) = \cos(n_{\mathrm{K}}y),\\
&\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} = -\frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y} = -\frac{\partial}{\partial y'},\\
&\frac{\partial^2}{\partial x^2} = -\frac{\partial}{\partial x'}\left(-\frac{\partial}{\partial x'}\right) = \frac{\partial^2}{\partial x'^{\,2}}, \qquad \frac{\partial^2}{\partial y^2} = -\frac{\partial}{\partial y'}\left(-\frac{\partial}{\partial y'}\right) = \frac{\partial^2}{\partial y'^{\,2}},
\end{aligned}$$

so that

$$\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x'^{\,2}} + \frac{\partial^2}{\partial y'^{\,2}} = \nabla'^{\,2},\\
\nabla^4 &= \nabla^2\,\nabla^2 = \nabla'^{\,2}\,\nabla'^{\,2} = \nabla'^{\,4}
\end{aligned}$$

and

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} = \left(-\frac{\partial f'}{\partial x'}\right)\left(-\frac{\partial g'}{\partial y'}\right) = \frac{\partial f'}{\partial x'}\,\frac{\partial g'}{\partial y'}$$

for two smooth functions f, g with the spatial symmetry $f = f'$ and $g = g'$.

Similarly, in the case of (9), we have

$$\cos(n_{\rm K} y') = \cos(n_{\rm K}\pi + n_{\rm K} y) = \cos(n_{\rm K} y),$$

since $n_{\rm K}$ is an even number, and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \qquad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}, \qquad \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial y'^2},$$

so that

$$\nabla^2 = \nabla'^2, \qquad \nabla^4 = \nabla'^4$$

and

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} = \frac{\partial f'}{\partial x'}\frac{\partial g'}{\partial y'}$$

for two smooth functions f,g with the spatial symmetry $f = f'$ and $g = g'$. This completes the proof of the lemma.

Lemma 2 (recursive symmetry). *For the two-dimensional Kolmogorov flow governed by the Navier–Stokes equation* (1) (*with an even* $n_{\rm K}$) *under the periodic boundary condition* (2) *subject to a smooth initial condition with the spatial symmetry of rotation* (3) *and/or translation* (4)*, if its solution* $\psi(x, y, t)$ *at* $t = t_0 \geqslant 0$ *possesses a spatial symmetry* $\psi(x, y, t_0) = \psi'(x', y', t_0)$ *of rotation and/or translation, then there exists the spatial symmetry* $\psi^{(m)}(x, y, t_0) = \psi'^{(m)}(x', y', t_0)$ *for all integers* $m \geqslant 1$.

Proof. The proof can be obtained by means of the recursive method.

Let $m = 1$. Since there exists the spatial symmetry at $t = t_0$, it holds

$$\psi'(x', y', t_0) = \psi(x, y, t_0), \quad \text{i.e.,} \quad \psi'^{(0)}(x', y', t_0) = \psi^{(0)}(x, y, t_0). \tag{12}$$

According to (11) and (12), we have at $t = t_0$ that

$$\begin{aligned}
\psi'^{(0)}_{x'}\nabla'^2\psi'^{(0)}_{y'} - \psi'^{(0)}_{y'}\nabla'^2\psi'^{(0)}_{x'} - \frac{1}{\rm Re}\nabla'^4\psi'^{(0)} = \\
&= \psi'^{(0)}_{x'}(\nabla'^2\psi'^{(0)})_{y'} - \psi'^{(0)}_{y'}(\nabla'^2\psi'^{(0)})_{x'} - \frac{1}{\rm Re}\nabla^4\psi^{(0)} = \\
&= \psi^{(0)}_{x}(\nabla^2\psi^{(0)})_{y} - \psi^{(0)}_{y}(\nabla^2\psi^{(0)})_{x} - \frac{1}{\rm Re}\nabla^4\psi^{(0)} = \\
&= \psi^{(0)}_{x}\nabla^2\psi^{(0)}_{y} - \psi^{(0)}_{y}\nabla^2\psi^{(0)}_{x} - \frac{1}{\rm Re}\nabla^4\psi^{(0)},
\end{aligned}$$

so that (10) becomes

$$\nabla^2\psi'^{(1)} + \psi^{(0)}_{x}\nabla^2\psi^{(0)}_{y} - \psi^{(0)}_{y}\nabla^2\psi^{(0)}_{x} - \frac{1}{\rm Re}\nabla^4\psi^{(0)} + n_{\rm K}\cos(n_{\rm K} y) = 0. \tag{13}$$

Comparing (13) with (7) yields the spatial symmetry at $t = t_0$:

$$\psi'^{(1)}(x', y', t_0) = \psi^{(1)}(x, y, t_0).$$

Thus, the lemma holds true when $m = 1$.

Assume that the lemma holds true when $m = 1, 2, \ldots, n$, i.e., there exists the spatial symmetry at $t = t_0$:

$$\psi'^{(i)}(x', y', t_0) = \psi^{(i)}(x, y, t_0), \qquad 0 \leqslant i \leqslant n. \tag{14}$$

Differentiating (1) and (10) n times with respect to t and then dividing by $n!$, we have

$$(n+1)\nabla^2\psi^{(n+1)} = \frac{1}{\mathrm{Re}}\nabla^4\psi^{(n)} - \sum_{i=0}^{n}[\psi_x^{(i)}\nabla^2\psi_y^{(n-i)} - \psi_y^{(i)}\nabla^2\psi_x^{(n-i)}] \tag{15}$$

in the coordinate system xOy, and

$$(n+1)\nabla'^2\psi'^{(n+1)} = \frac{1}{\mathrm{Re}}\nabla'^4\psi'^{(n)} - \sum_{i=0}^{n}[\psi'^{(i)}_{x'}\nabla'^2\psi'^{(n-i)}_{y'} - \psi'^{(i)}_{y'}\nabla'^2\psi'^{(n-i)}_{x'}] \tag{16}$$

in the coordinate system $O' - x'y'$, respectively.

According to (11) and (14), we have at $t = t_0$ that

$$\nabla'^2\psi'^{(n+1)} = \nabla^2\psi'^{(n+1)}, \qquad \nabla'^4\psi'^{(n)} = \nabla^4\psi^{(n)}, \tag{17}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n}[\psi'^{(i)}_{x'}\nabla'^2\psi'^{(n-i)}_{y'} - \psi'^{(i)}_{y'}\nabla'^2\psi'^{(n-i)}_{x'}] = \\
&\qquad = \sum_{i=0}^{n}[\psi'^{(i)}_{x'}(\nabla'^2\psi'^{(n-i)})_{y'} - \psi'^{(i)}_{y'}(\nabla'^2\psi'^{(n-i)})_{x'}] = \\
&\qquad = \sum_{i=0}^{n}[\psi_x^{(i)}(\nabla^2\psi^{(n-i)})_y - \psi_y^{(i)}(\nabla^2\psi^{(n-i)})_x] = \\
&\qquad = \sum_{i=0}^{n}[\psi_x^{(i)}\nabla^2\psi_y^{(n-i)} - \psi_y^{(i)}\nabla^2\psi_x^{(n-i)}].
\end{aligned} \tag{18}$$

Substituting (17) and (18) into (16) gives

$$(n+1)\nabla^2\psi'^{(n+1)} = \frac{1}{\mathrm{Re}}\nabla^4\psi^{(n)} - \sum_{i=0}^{n}\left[\psi_x^{(i)}\nabla^2\psi_y^{(n-i)} - \psi_y^{(i)}\nabla^2\psi_x^{(n-i)}\right]. \tag{19}$$

Comparing (19) with (15) yields the spatial symmetry

$$\psi'^{(n+1)}(x', y', t_0) = \psi^{(n+1)}(x, y, t_0)$$

for any integer $n \geqslant 1$. Thus, if the lemma holds for an integer $n \geqslant 1$, it also holds true for the integer $n+1$.

Due to the recursive method, we have that the spatial symmetry

$$\psi'^{(m)}(x', y', t_0) = \psi^{(m)}(x, y, t_0), \tag{20}$$

holds for an arbitrary integer $m \geqslant 1$. This completes the proof of the lemma.

Lemma 3 (symmetry continuation). *For the two-dimensional Kolmogorov flow governed by the Navier–Stokes equation* (1) (*with an even* n_K) *under the periodic boundary condition* (2) *subject to a smooth initial condition with the spatial symmetry of rotation* (3) *and/or translation* (4), *if its solution* $\psi(x, y, t)$ *has a spatial symmetry at* $t = t_0$, *say,* $\psi'(x', y', t_0) = \psi(x, y, t_0)$, *and furthermore, if its Taylor series exists at* $t = t_0$ *with a nonzero radius* ρ *of convergence, then* $\psi(x, y, t)$ *in* $t \in [t_0, t_0 + \rho)$ *maintains the same spatial symmetry as* $\psi(x, y, t_0)$.

Proof. According to Lemma 2, from the spatial symmetry $\psi'(x', y', t_0) = \psi(x, y, t_0)$, we have the spatial symmetry

$$\psi'^{(m)}(x', y', t_0) = \psi^{(m)}(x, y, t_0), \qquad 1 \leqslant m < +\infty.$$

Then,

$$\psi'(x', y', t) = \sum_{m=0}^{+\infty} \psi'^{(m)}(x', y', t_0)\ (t - t_0)^m = \sum_{m=0}^{+\infty} \psi^{(m)}(x, y, t_0)(t - t_0)^m = \psi(x, y, t),$$

where $|t - t_0| < \rho$, including $t \in [t_0, t_0 + \rho)$, with $\rho > 0$ being the radius of convergence. This guarantees the spatial symmetry $\psi'(x', y', t) = \psi(x, y, t)$ in the interval $t \in [t_0, t_0 + \rho)$. This completes the proof of the lemma.

Theorem 1 (spatial symmetry invariance). *For the* 2*D Kolmogorov flow governed by the Navier–Stokes equation* (1) (*with an even* n_K) *under the periodic boundary condition* (2) *subject to a smooth initial condition with the spatial symmetry of rotation* (3) *and/or translation* (4), *if its Taylor series exists with a nonzero convergence radius* ρ *for all* $t \geqslant 0$, *its solution for all* $t \geqslant 0$ *maintains the same spatial symmetry as its initial condition.*

Proof. Since the initial condition contains the spatial symmetry $\psi(x, y, 0) = \psi'(x', y', 0)$, we have according to Lemma 3 (by setting $t_0 = 0$) the spatial symmetry:

$$\psi(x, y, t) = \psi'(x', y', t), \qquad t \in [0, \delta_0],$$

where $\delta_0 < \rho_0$, and ρ_0 is the radius of convergence of its Taylor series at $t = 0$. Then, setting $t_0 = \delta_0$, we have according to Lemma 3 the spatial symmetry:

$$\psi(x, y, t) = \psi'(x', y', t), \qquad t \in [\delta_0, \delta_1],$$

where $\delta_1 < \rho_1$, and ρ_1 is the radius of convergence of its Taylor series at $t = t_0 = \delta_0$. Furthermore, setting $t_0 = \delta_1$, we have according to Lemma 3 the spatial symmetry:

$$\psi(x, y, t) = \psi'(x', y', t), \qquad t \in [\delta_1, \delta_2],$$

where $\delta_2 < \rho_2$, and ρ_2 is the radius of convergence of its Taylor series at $t = t_0 = \delta_1$. Note that

$$[0, \delta_0] \cup [\delta_0, \delta_1] \cup [\delta_1, \delta_2] = [0, \delta_2],$$

which means that now there exists the spatial symmetry

$$\psi(x, y, t) = \psi'(x', y', t), \qquad t \in [0, \delta_2],$$

in a larger interval of time. The same process can be continued, since the Taylor series of $\psi(x, y, t)$ always exists with a nonzero radius of convergence. Thus, it always holds that the spatial symmetry persists for all $t \in [0, +\infty)$:

$$\psi(x, y, t) = \psi'(x', y', t), \qquad t \in [0, +\infty).$$

This completes the proof of the theorem. Note that it is easy to prove a similar theorem for the three-dimensional NS equations in general (see the Appendix of [37]).

3. Remarks and discussions

Remark 1. Let O denote one observer in the coordinate system xOy, and let O' denote another observer in the coordinate system $O' - x'y'$, respectively, where $(x', y') = (x + \pi, y + \pi)$ and/or $(x', y') = (2\pi - x, 2\pi - y)$. When there exists the spatial symmetry $\psi'(x', y', 0) = \psi(x, y, 0)$ for the initial condition, the governing equations and the initial condition are the *same* for both the observer O in the coordinate system xOy and the observer O' in the coordinate system $O' - x'y'$. Thus, physically speaking, the two observers in the two different coordinate systems should observe the same flow. From the viewpoint of a third observer simultaneously knowing the relationship between their coordinate systems and their results, the flow governed by the NS equations (1) should keep the same spatial symmetry as its initial condition. This physical insight is indeed true, as mathematically proved in this paper. Thus, the theorem of spatial symmetry invariance is easy to understand from a physical viewpoint. Note that, based on the covariance of the NS equations, Kida [38] assumed that a turbulent flow should maintain the same spatial symmetry as the initial condition, and thus used spatial symmetry to reduce the simulation domain by 1/64 so as to achieve higher Reynolds numbers with limited resources: it treated the spatial symmetry as an observed utility rather than a proven law, although in practice DNS violates the theorem of spatial symmetry invariance.

Remark 2. Numerical simulations of the 2D Kolmogorov flow governed by the Navier–Stokes equation (1) (with an even $n_{\rm K}$) under the periodic boundary condition (2) subject to a smooth initial condition with the spatial symmetry of rotation (3) and/or translation (4) must distinctly deviate from the exact solution, if they violate the spatial symmetry invariance theorem proved in this paper. Thus, this theorem can be used as "an absolute scale" to check the correctness and accuracy of numerical simulations of the 2D turbulent Kolmogorov flow.

For example, as reported in [31], [32], the numerical simulations of the 2D turbulent Kolmogorov flow given by CNS (in a finite but long enough time-interval) always maintain the same spatial symmetry as the corresponding initial condition. This agrees well with the theorem of spatial symmetry invariance. This is mainly because, unlike the traditional algorithms, numerical noises of CNS are rigorously negligible in a finite time-interval that is, however, long enough for calculating statistics. However, as mentioned in [31], [32], the corresponding DNS results retain the same spatial symmetry as the corresponding initial condition only for a short duration at the beginning, but then quickly lose it, clearly indicating that the DNS results distinctly deviate from the exact solution of the NS equations. In other words, these DNS results violate the theorem of spatial symmetry invariance. This clearly indicates that the DNS results are indeed quickly polluted by numerical noises badly, which are stochastic and thus certainly have no spatial symmetry at all. The above conclusions should have general meaning for turbulent flows, as illustrated in [30].

Remark 3. For the 2D Kolmogorov flow governed by the Navier–Stokes equation (1) (with an even $n_{\rm K}$) under the periodic boundary condition (2) subject to a smooth initial condition with the spatial symmetry of rotation (3) and/or translation (4), if the initial condition has different spatial symmetries, their corresponding solutions should be distinctly different even in their statistics.

For example, Liao and Qin [32] applied CNS to solve the 2D turbulent Kolmogorov flow (1) in the case of $\mathrm{Re} = 2000$ and $n_{\rm K} = 16$, subject to the following different initial conditions:

$$\psi_1(x, y, 0) = -\frac{1}{2}[\cos(x + y) + \cos(x - y)], \tag{21}$$

$$\psi_2(x, y, 0) = -\frac{1}{2}[\cos(x + y) + \cos(x - y)] + \delta' \sin(x + y), \tag{22}$$

where δ' is a constant. Note that $\psi_1(x, y, 0)$ contains the spatial symmetry of both rotation and translation, but $\psi_2(x, y, 0)$ contains the spatial symmetry of only translation for a large $\delta' \sim O(1)$. As reported by Liao and Qin [32], the spatiotemporal trajectories given by CNS using the initial condition $\psi_1(x, y, 0)$ retain the same spatial symmetry of rotation and translation as the initial condition $\psi_1(x, y, 0)$ in the whole time-interval $t \in [0, 300]$. This agrees well with the theorem of spatial symmetry invariance proved in Sec. 2.

When $\delta' = 10^{-20}$ in (22), the spatiotemporal trajectories given by CNS using the initial condition $\psi_2(x, y, 0)$ seem to remain in the same spatial symmetry of rotation and translation as the initial condition $\psi_1(x, y, 0)$ at the beginning, mainly because the disturbance $10^{-20} \sin(x + y)$ of $\psi_2(x, y, 0)$ is too small compared to $\psi_1(x, y, 0)$. However, until it enlarges to a macro-level, due to the so-called "butterfly-effect" of the chaotic characteristics of turbulence [11]–[15], at about $t \approx 35$, it finally destroys the original spatial symmetry of $\psi_1(x, y, 0)$.

Thus, the CNS result thereafter remains the same spatial symmetry of translation as its initial condition $\psi_2(x, y, 0)$ for large $\delta' \sim O(1)$, as reported by Liao and Qin [32]. This CNS result also agrees well with the theorem of spatial symmetry invariance. Certainly, the two CNS results should have different statistics, since they have different spatial symmetries. Thus, the theorem of spatial symmetry invariance also supports the conclusion that the turbulence is a kind of ultra-chaos [29], [39], i.e., a small disturbance could lead to distinct differences not only in the spatotemporal trajectory but also in the statistics of turbulence.

It should be emphasized again that, the corresponding DNS results of the same turbulent Kolmogorov flows quickly lose their spatial symmetry, since stochastic numerical noises without spatial symmetry quickly enlarge to a macro-level. Therefore, the theorem of spatial symmetry invariance rigorously confirms that the DNS results of the 2D turbulent Kolmogorov flow are indeed quickly polluted by numerical noises badly, as illustrated in [31]. This is also the reason why the so-called "noise-expansion cascade" revealed in [32] by means of CNS cannot be discovered by means of DNS. This is a good example to illustrate that CNS can reveal the truth and reality of NS turbulence indeed.

For a large enough Reynolds number such as $\mathrm{Re} = 2000$ considered in [32], the 2D Kolmogorov flow (1) is turbulent. It is widely accepted by fluid community that turbulence is chaotic [11]–[15] and thus, due to the famous butterfly-effect of a chaotic system, the small disturbance $\delta' \sin(x + y)$ of the initial condition $\psi_2(x, y, t)$ defined by (22) should be quickly enlarged to the same order of magnitude as the first initial condition $\psi_1(x, y, t)$ described by (21). However, $\psi_1(x, y, t)$ has a different spatial symmetry from that of $\psi_2(x, y, t)$ when $\delta' \sim O(1)$; therefore, according to the theorem of spatial symmetry invariance, the turbulent Kolmogorov flow with the initial condition $\psi_1(x, y, t)$ has a different spatial symmetry from that with $\psi_2(x, y, t)$ even for an arbitrarily small δ', say, $\delta' \to 0$. Thus, the theorem of spatial symmetry invariance plus the butterfly-effect of chaos supports the following conjecture.

Conjecture 1. *The two-dimensional turbulent Kolmogorov flow governed by the NS equation* (1) (*with an even* n_{K}) *under the periodic boundary condition* (2) *subject to a smooth initial condition with the spatial symmetry of rotation* (3) *and/or translation* (4) *admits distinct smooth global solutions for all* $t \geqslant 0$ *from almost the same initial conditions* $\psi_1(x, y, 0)$ *and* $\psi_2(x, y, 0)$ *with* $|\psi_1(x, y, 0) - \psi_2(x, y, 0)| \leqslant \delta$ *for an arbitrarily small* δ *(or as* $\delta \to 0$*).*

As reported by Qin and Liao [33], the CNS results of the three-dimensional (3D) Kolmogorov flow also maintain the same spatial symmetry as its initial condition in the whole simulation interval of time $t \in [0, 500]$. This highly suggests that Conjecture 1 should have general meaning. All of these support the following conjecture mentioned by Liao in [40]:

Conjecture 2. *The Navier–Stokes turbulence admits distinct smooth global solutions* $\mathbf{u}_1(\mathbf{r}, t)$ *and* $\mathbf{u}_2(\mathbf{r}, t)$ *for all* $t > 0$ *from almost the same initial conditions* $\mathbf{u}_1(\mathbf{r}, 0)$ *and* $\mathbf{u}_2(\mathbf{r}, 0)$ *with* $|\mathbf{u}_1(\mathbf{r}, 0) - \mathbf{u}_2(\mathbf{r}, 0)| \leqslant \delta$ *for arbitrarily small* δ *(or as* $\delta \to 0$*).*

Obviously, if the flow is laminar, any small disturbance in the initial condition will not increase to a macro-level, but rather remain small globally in time. Thus, the above two conjectures should hold true only for turbulent flows. Finally, it should be emphasized that the above two conjectures might have close relationships with the nonuniqueness of solutions to the NS equations, which belongs to the Millennium Problems of the Clay Mathematics Institute (see https://www.claymath.org/millennium-problems/).

What happens when the initial condition has no spatial symmetry? Qin and Liao [30] applied both of DNS and CNS to solve the 2D Rayleigh–Bénard turbulent flow using a thermal fluctuation as the initial condition, which is random in space and thus has no spatial symmetry: it was found [30] that the flow given by CNS remains a kind of vortical flow in the whole duration $t \in [0, 500]$, but the flow given by DNS is first a kind of vortical flow from the beginning of the simulation, until $t \approx 188$ when it transitions into a kind of zonal flow thereafter in $t \in [188, 500]$. Currently, it has been found that the final flow type (given by the DNS) of the 2D Rayleigh–Bénard turbulent flow using a random thermal fluctuation as the initial condition are very sensitive to the time-step for some Reynolds numbers [41]: the two flow types frequently alternate as the time-step is reduced to a very small value, suggesting that the time-step corresponding to each turbulent flow type is densely distributed. All of these examples clearly illustrate that numerical noises indeed have a huge influence on the global characteristics of turbulence, at least in some cases; it remains an open question whether or not a stochastic initial condition must be used for NS turbulence.

Note that, unlike the exact solution mentioned in the theorem of spatial symmetry invariance proved in Sec. 2, the numerical results given by CNS are reliable only in a finite but long enough interval of time $t \in [0, T_c]$, where T_c is called the "critical predictable time." As pointed out by Liao in his book [22], $T_c \sim -\kappa \ln \mathcal{E}_0$ for a chaotic system and turbulence, where $\mathcal{E}_0 > 0$ denotes the level of background numerical noises, and $\kappa > 0$ is the Lyapunov exponent of the chaotic system. Obviously, the smaller the background numerical noise $\mathcal{E}_0$, the larger the so-called "critical predictable time" T_c. Since $\mathcal{E}_0$ of CNS is much smaller than that of DNS, CNS can give reliable simulations of chaotic systems and turbulent flows in a much larger time-interval $t \in [0, T_c]$.

From this viewpoint, we can regard DNS as a special case of CNS, because DNS often uses single or double precision with the 4th-order Runge–Kutta method in the time dimension, corresponding to a rather small "critical predictable time" T_c that is too short for statistical computation. It is a pity that DNS *lacks* the concept of "critical predictable time," since DNS assumes, wrongly, that the numerical noises of NS turbulence should not increase to a macro-level.

It should be emphasized that, although the CNS results are numerical, they can provide us with the insights needed to approach some mathematical truths and realities of NS turbulence, such as the noise-expansion cascade [32], the ultra-chaotic property [39], the theorem of spatial symmetry invariance proved in Sec. 2, and the conjectures mentioned above and in [40]. Hopefully, CNS [21]–[33], [37], [39], [40] could be used as a powerful tool to study turbulence, although further research is necessary in the future.

Note that the NS equations are deterministic, since all stochastic small disturbances are neglected. However, this is only an assumption; thus, it is still an open question [41] whether this assumption is physically reasonable or not.

Why do many DNS results of NS turbulence agree well with experimental data, even if they are badly polluted by numerical noises? This is because there exist two types of turbulent flows, i.e., normal-chaos and ultra-chaos: the statistics of an ultra-chaos are very sensitive to disturbances, but those of a normal-chaos are not. For example, the chaotic solution of the Lorenz equation is a normal-chaos: although its trajectories given by traditional algorithms are badly polluted by numerical noises and greatly depart from its exact solution, its statistics are stable and agree well with the CNS result, as reported in [23]. Thus, if a turbulent flow is a normal-chaos, its statistics should be stable and thus could agree well with experimental data, even if its spatiotemporal trajectories are badly polluted by numerical noises. Unfortunately, NS turbulence is ultra-chaotic in essence, as mentioned in [39]. It should be emphasized that the exact solution of NS turbulence is one thing, and the numerical simulation of NS turbulence is another thing: both of them are mathematical entities and thus are essentially different from practical turbulence in physics, as discussed in [39].

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REFERENCES

1. E. N. Lorenz, "Deterministic nonperiodic flow," *J. Atmos. Sci*, **20**, 130–141 (1963).
2. S. Smale, "Differentiable dynamical systems," *Bull. Amer. Math. Soc.*, **73**, 747–817 (1967).
3. T.-Y. Li and J. A. Yorke, "Period three implies chaos," *Amer. Math. Monthly*, **82**, 985–992 (1975).
4. R. M. May, "Simple mathematical models with very complicated dynamics," *Nature*, **261**, 459–467 (1976).
5. M. J. Feigenbaum, "Quantitative universality for a class of nonlinear transformations," *J. Stat. Phys.*, **19**, 25–52 (1978); "The universal metric properties of nonlinear transformations," **21**, 669–706 (1979).
6. E. N. Lorenz, "Computational periodicity as observed in a simple system," *Tellus A*, **58**, 549–557 (2006).
7. C. L. M. H. Navier, "Mémoire sur le lois du mouvement des fluides," *Mém. Acad. Roy. Sci. Paris*, **6**, 389–416 (1823).
8. G. G. Stokes, "On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids," *Trans. Cambridge Philos. Soc.*, **8**, 287–305 (1845).
9. O. Reynolds, "On the dynamical theory of turbulent incompressible viscous fluids and the determination of the criterion," *Phil. Trans. R. Soc. London Ser. A*, **186**, 123–164 (1895).
10. G. I. Taylor, "Statistical theory of turbulence," *Proc. R. Soc. London Ser. A*, **151**, 421–444 (1935).
11. R. G. Deissler, "Is Navier–Stokes turbulence chaotic?," *Phys. Fluids*, **29**, 1453–1457 (1986).
12. J. Ge, J. Rolland, and J. C. Vassilicos, "The production of uncertainty in three-dimensional Navier–Stokes turbulence," *J. Fluid Mech.*, **977**, A17, 33 pp. (2023).
13. G. Boffetta and S. Musacchio, "Predictability of the inverse energy cascade in 2D turbulence," *Phys. Fluids*, **13**, 1060–1062 (2001).
14. G. Boffetta and S. Musacchio, "Chaos and predictability of homogeneous-isotropic turbulence," *Phys. Rev. Lett.*, **119**, 054102, 5 pp. (2017).
15. A. Berera and R. D. J. G. Ho, "Chaotic properties of a turbulent isotropic fluid," *Phys. Rev. Lett.*, **120**, 024101, 5 pp. (2018).
16. S. A. Orszag, "Analytical theories of turbulence," *J. Fluid Mech.*, **41**, 363–386 (1970).
17. Z.-S. She, E. Jackson, and S. A. Orszag, "Structure and dynamics of homogeneous turbulence: models and simulations," *Proc. R. Soc. London Ser. A*, **434**, 101–124 (1991).
18. K. Moin and K. Mahesh, "Direct numerical simulation: A tool in turbulence research," *Annu. Rev. Fluid Mech.*, **30**, 539–578 (1998).
19. R. Scardovelli and S. Zaleski, "Direct numerical simulation of free-surface and interfacial flow," *Annu. Rev. Fluid Mech.*, **31**, 567–603 (1999).
20. G. N. Coleman and R. D. Sandberg, A primer on direct numerical simulation of turbulence—methods, procedures and guidelines (Tech. Rep. AFM-09/01a), Aerodynamics & Flight Mechanics Research Group, University of Southampton, UK (2010).
21. S. Liao, "On the reliability of computed chaotic solutions of non-linear differential equations," *Tellus A*, **61**, 550–564 (2009).
22. S. Liao, *Clean Numerical Simulation*, Advances in Applied Mathematics, Chapman and Hall/CRC, Boca Raton, FL (2023).
23. S. Liao and P. Wang, "On the mathematically reliable long-term simulation of chaotic solutions of Lorenz equation in the interval $[0, 10000]$," *Sci. China Phys. Mech. Astron.*, **57**, 330–335 (2014).

24. X. Li and S. Liao, "More than six hundred new families of Newtonian periodic planar collisionless three-body orbits," *Sci. China Phys. Mech. Astron.*, **60**, 129511, 7 pp. (2017).
25. X. Li, Y. Jing, and S. Liao, "Over a thousand new periodic orbits of a planar three-body system with unequal masses," *Publ. Astron. Soc. Japan*, **70**, 64, 7 pp. (2018).
26. S. Liao, X. Li, and Y. Yang, "Three-body problem—from Newton to supercomputer plus machine learning," *New Astron.*, **96**, 101850, 11 pp. (2022).
27. T. Hu and S. Liao, "On the risks of using double precision in numerical simulations of spatio-temporal chaos," *J. Comput. Phys.*, **418**, 109629, 25 pp. (2020).
28. S. Qin and S. Liao, "Influence of numerical noises on computer-generated simulation of spatio-temporal chaos," *Chaos Solitons Fractals*, **136**, 109790, 13 pp. (2020).
29. S. Liao and S. Qin, "Ultra-chaos: An insurmountable objective obstacle of reproducibility and replication," *Adv. Appl. Math. Mech.*, **14**, 799–815 (2022).
30. S. Qin and S. Liao, "Large-scale influence of numerical noises as artificial stochastic disturbances on a sustained turbulence," *J. Fluid Mech.*, **948**, A7, 25 pp. (2022).
31. S. Qin, Y. Yang, Y. Huang, X. Mei, L. Wang, and S. Liao, "Is a direct numerical simulation (DNS) of Navier–Stokes equations with small enough grid spacing and time-step definitely reliable/correct?," *J. Ocean Eng. Sci*, **9**, 293–310 (2024).
32. S. Liao and S. Qin, "Noise-expansion cascade: an origin of randomness of turbulence," *J. Fluid Mech.*, **1009**, A2, 22 pp. (2025).
33. S. Qin and S. Liao, "Clean numerical simulation of three-dimensional turbulent Kolmogorov flow," *Phys. Fluids*, **37**, 105160, 15 pp. (2025).
34. A. M. Obukhov, "Kolmogorov flow and laboratory simulation of it," *Russian Math. Surveys*, **38**, 113–126 (1983).
35. G. J. Chandler and R. R. Kerswell, "Invariant recurrent solutions embedded in a turbulent two-dimensional Kolmogorov flow," *J. Fluid Mech.*, **722**, 554–595 (2013).
36. W. Wu, F. G. Schmitt, E. Calzavarini, and L. Wang, "A quadratic Reynolds stress development for the turbulent Kolmogorov flow," *Phys. Fluids*, **33**, 125–129 (2021).
37. S. Liao, "Spatial symmetry invariance of solution of Kolmogorov flow," arXiv: 2603.04447.
38. S. Kida, "Numerical simulation of two-dimensional turbulence with high-symmetry," *J. Phys. Soc. Japan*, **54**, 2840–2854 (1985).
39. S. Qin, K. Xu, and S. Liao, "Ultra-chaotic property of Navier–Stokes turbulence," *Eur. J. Mech. – B/Fluids*, **119**, 204540, 7 pp. (2026), arXiv: 2510.06620.
40. S. Liao and S. Qin, "Non-uniqueness of smooth solutions of the Navier–Stokes equations from almost the same initial conditions," arXiv: 2602.12666.
41. S. Qin, K. Xu, and S. Liao, "A paradox of the Navier–Stokes turbulence," arXiv: 2510.11220.

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