

A review on periodic orbits of the general planar three-body problem

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The three-body problem continues to be an unsolved challenge in the scientific community. It serves as a foundational model for exploring nonlinear dynamics, astrophysics, and orbital mechanics. Due to the inherently chaotic nature of the three-body problem, periodic orbits are regarded as the primary framework that forms the backbone of the system, which provides crucial insights into its dynamical properties. In this review, we examine both historical and recent advances in the study of periodic orbits for the planar three-body problem. We provide an overview of the general three-body problem, the definition and identification of numerical periodic orbits, and the topological classification methods used to categorize these solutions. Additionally, we describe the clean numerical simulation (CNS) employed to obtain accurate trajectories, as well as both analytical and numerical results for periodic orbits in systems with different initial configurations, with equal or unequal masses, with or without angular momentum. The review also covers key topics such as linear stability analysis and the extension of Kepler's third law to general three-body problems. Finally, we highlight several open questions and promising directions for future research in this field.

planar three-body problem, periodic orbits, clean numerical simulation, linear stability

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1 Introduction

The three-body problem is a fundamental challenge in mathematics and physics. It describes the motion of three bodies under Newtonian gravitational attraction. Understanding the three-body problem is crucial for gaining insights into the more general n -body problem, particularly regarding the long-term evolution and stability of celestial systems [1]. The three-body problem is relevant to many astrophysical phenomena such as binary star formation [2], the Sun-Jupiter-Trojan system [3], the chaotic dynamics of main belt asteroids [4], and mean-motion resonance in planetary systems [5]. Additionally, the restricted three-body problem serves as

an ideal model for designing spacecraft trajectories in deep space exploration [6].

The origins of the three-body problem can be traced back to Newton's Principia in 1687, where he analyzed the Moon's motion as influenced by both the Sun and the Earth [7]. Euler [8] later presented the collinear solution and Lagrange [9] provided the equilateral solution. In the Euler's solution, the three bodies remain collinear at all times. For the Lagrange's solution, the three bodies form an equilateral triangle at all times while moving around their common center of mass in elliptical orbits. Euler also formulated the circular restricted three-body problem in a rotating frame and presented three equilibrium points, and Lagrange added another two. The first observational confirmation of the Lagrange's solution in

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celestial mechanics was documented in 1906 through the discovery of Trojan asteroids co-orbiting with Jupiter around the Sun.

As is well known, the two-body problem has a closed-form solution. However, the problem becomes significantly more complex when a third body is introduced, leading to the famous three-body problem. Bruns [10] demonstrated that no further integrals exist for the three-body problem, meaning that it can not be solved in the same manner as the two-body problem. Poincaré [11] was the first to demonstrate the initial sensitivity of the three-body problem, a phenomenon that later became widely known as chaos. He also emphasized the importance of periodic orbits, stating that “what makes these (periodic) solutions so precious to us, is that they are, so to say, the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible.” Thus, periodic orbits provide a fundamental framework for understanding the dynamical behavior of three-body systems.

In the early 20th century, Burrau [12] numerically investigated the Pythagorean problem, where three bodies with masses 3, 4, and 5 are positioned at the vertices of a right-angled triangle with sides length 3, 4, and 5. The numerical results show that the system eventually disrupts, leading to the formation of a close binary system and a single escaping body. By the mid-20th century, there was considerable debate about the existence of periodic solutions for the general three-body problem beyond the well-known Lagrange solutions. Vernic [13] argued that no such solutions existed, but this claim was challenged by Merman [14] and ultimately refuted by Arenstorf [15], who proved the existence of periodic orbits, though without providing specific examples.

The advent of electronic computers revolutionized this field, enabling extensive numerical exploration of the three-body problem. Szebehely and Peters [16] employed numerical methods to find periodic solutions in the free-fall three-body problem involving two-body collisions. Subsequently, a non-colliding periodic solution was later discovered, confirming the existence of periodic solutions beyond the Lagrange’s solutions [17]. During the 1970s, many researchers investigated periodic orbits in the three-body problem using numerical methods. These studies focused on configurations with collinear initial conditions in either a fixed or rotating frame [18–20]. In 1993, Moore [21] numerically obtained a special periodic orbit for the three-body problem with equal masses, where the three bodies move along the same trajectory in the shape of a figure-eight. The figure-eight orbit was later rediscovered and rigorously proven by Chenciner and Montgomery [22] using a variational method.

For the three-body problem, Montgomery [23] proposed a topological method to classify their trajectories by map-

ping the orbits onto a “shape sphere”. Using this topological classification method [23], three main families of periodic solutions were known for the three-body system over 300 years: (1) Euler-Lagrange family [8, 9], (2) Broucke-Hadjidemetriou-Hénon (BHH) family [18–20], and (3) figure-eight family [21, 22]. This changed in 2013 when Šuvakov and Dmitrašinović [24] presented 13 new distinct periodic orbits belonging to 11 families. For the figure-eight family of periodic orbits, it was demonstrated that the topological properties are closely related to energy and period [25]. Additionally, the relationship between angular momentum, topological properties, and Kepler’s third law was investigated for the BHH family of periodic three-body orbits [26]. It was also found that the inverse of total energy approximately follows a power law with an exponent of $2/3$ with respect to topological average period for massive periodic orbits [27]. This observation suggests a quasi Kepler’s third law for the three-body problem.

It is well known that the three-body problem is inherently chaotic, meaning that small perturbations in initial conditions can lead to significantly different outcomes [11]. This sensitivity makes reliable numerical simulations crucial for accurately computing periodic solutions. Liao [28] introduced clean numerical simulation (CNS) to obtain reliable solutions for chaotic Lorenz equations, and this method was later extended to the three-body problem [29]. Recently, the CNS has been employed to find thousands of new periodic orbits for the three-body system [27, 30–32]. Li et al. [32] also utilized the CNS to obtain 1646 new triple collision orbits without binary collisions for the three-body system with zero initial velocities.

Over the past decade, numerical searches for periodic orbits in the general three-body problem have received significant attention. For satellite orbits of the BHH family, Janković and Dmitrašinović [26] presented 57 periodic orbits with different angular momentum. More details on the numerical search method, along with 100 periodic orbits, were provided in ref. [33]. For the three-body problem with zero initial velocities, Yasko and Orlov [34] searched for approximate periodic orbits of the equal-mass three-body system within the Agekyan and Anosova’s initial region [35]. The CNS was used to obtain 313 new collisionless periodic free-fall three-body orbits [31]. Recently, this number has been expanded to 12409 distinct solutions [36]. Numerical studies were also conducted for systems with three equal-mass bodies [24, 37, 38], two equal-mass bodies [39], and systems with different-mass bodies [32]. Furthermore, the stability of these periodic three-body orbits was also investigated in detail [32, 40, 41]. Thus, periodic orbits of the general three-body problem remain an active area of research. However, review papers on periodic three-body orbits have mainly fo-

cused on the restricted three-body problem [6] where the mass of one body is negligible, such as resonant periodic orbits [42]. Therefore, it is necessary to review both the historical and current developments in the study of periodic orbits in the general three-body problem.

This paper is organized as follows: in Section 2 we present the equations of motion for the general three-body problem and define periodic orbits. Section 3 describes the topological methods used for classifying these orbits. Analytical periodic orbits and numerical methods are introduced in Sections 4 and 5, respectively. In Section 6, we provide a detailed overview of numerical periodic three-body orbits. Linear stability and Kepler's third law are explored in Sections 7 and 8. Finally, discussion and conclusion are presented in Section 9.

2 Equations of motion for the general planar three-body problem and definition of periodic orbits

It is assumed that the motion of the three bodies is confined to a plane. Under this assumption, the motion of the Newtonian planar three-body system in the Cartesian coordinate system can be described by the following second-order differential equations:

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j \neq i}^3 \frac{Gm_j(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \quad i = 1, 2, 3, \quad (1)$$

where $\mathbf{r}_i = (x_i, y_i)$ represents the two-dimensional coordinate of the i -th body, m_i is its mass, and G is the Newtonian gravity constant. Without loss of generality, the Newtonian gravity constant G is set to unity. The state of the three-body system at any time t can be represented by a vector $\mathbf{X}(t)$, which includes positions and velocities of the three bodies: $\mathbf{X}(t) = (\mathbf{r}_1(t), \mathbf{r}_2(t), \mathbf{r}_3(t), \dot{\mathbf{r}}_1(t), \dot{\mathbf{r}}_2(t), \dot{\mathbf{r}}_3(t))$, with the initial condition $\mathbf{X}(0)$ specifying the system's starting configuration.

Periodic orbits in the three-body problem come in two primary types: absolute periodic orbits and relative periodic orbits. In the case of absolute periodic orbits, the system returns exactly to its initial state after a period T , that is, $|\mathbf{X}(T) - \mathbf{X}(0)| = 0$ in a fixed reference frame. For relative periodic orbits, the mutual distances between the three bodies are periodic functions of time. This means that after a period T , the system returns to its initial configuration, but in a rotating reference frame [43].

Since there is no general analytical solution for the three-body problem, periodic orbits are typically defined numerically. A numerical solution $\mathbf{X}(t)$ is considered periodic if, for

a given tolerance $\epsilon > 0$, there exist an initial condition $\mathbf{X}(0)$ and a period T such that $|\mathbf{X}(T) - \mathbf{X}(0)| < \epsilon$. To quantify how closely the system returns to its initial state after one period, the return proximity function $d(\mathbf{X}(0), T)$ is defined to measure the distance between the state at $t = 0$ and $t = T$ as follow [44]:

$$d(\mathbf{X}(0), T) = |\mathbf{X}(T) - \mathbf{X}(0)| \quad (2)$$

$$= \sqrt{\sum_{i=1}^3 [\mathbf{r}_i(T) - \mathbf{r}_i(0)]^2 + \sum_{i=1}^3 [\dot{\mathbf{r}}_i(T) - \dot{\mathbf{r}}_i(0)]^2}.$$

This function serves as a critical measure for evaluating the periodicity of numerical solutions in the three-body problem.

3 Topological classification of periodic orbits of the planar three-body problem

Classifying periodic orbits provides a framework for understanding the dynamical properties of the planar three-body problem. In the restricted three-body problem, simpler orbits can be categorized based on the motion of an infinitesimally small body around two larger bodies [45]. However, this approach becomes insufficient for the more complex orbits encountered in the general three-body problem. To address this, Moore [21] introduced braid theory to describe periodic orbits in the classical Newtonian N -body problem. Montgomery [23] also proposed a topological method to classify three-body orbits, which has since become widely used to categorize periodic solutions in the three-body problem [24, 27]. This topological classification method focuses on the sequence of syzygies-configurations where the three bodies become collinear.

To visualize these configurations, the relative positions of three bodies are mapped onto a "shape sphere" [23]. This is achieved by representing the triangle formed by the bodies with a unit vector $\mathbf{n}$,

$$\mathbf{n} = (n_x, n_y, n_z) = \left(\frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{R^2}, \frac{\lambda^2 - \rho^2}{R^2}, \frac{2(\boldsymbol{\rho} \times \boldsymbol{\lambda}) \cdot \mathbf{e}_z}{R^2} \right), \quad (3)$$

where $\mathbf{e}_z$ denotes the unit vector along the z -axis, and the two relative vectors $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ are defined as

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2), \quad \boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3), \quad (4)$$

and $R = \sqrt{\rho^2 + \lambda^2}$ is the normalization factor. Note that $\mathbf{n}$ represents a three-dimensional unit vector, which lies on the surface of a sphere with unit radius, as illustrated in Figure 1. Each point on this shape sphere corresponds to a unique shape of the triangle formed by the three bodies, independent

of its size or orientation. The equator of the sphere represents collinear configurations (syzygies), with three specific points marking two-body collisions. The poles of the sphere correspond to equilateral triangle configurations. A periodic orbit traces a closed curve on this sphere, as shown in Figure 1.

For further analysis, the shape sphere can be projected onto a two-dimensional plane, mapping one of the binary collision points to infinity and leaving two binary collision points in the finite plane, as shown in Figure 2. The resulting closed curve on the plane characterizes the topological properties of the orbit. These planar curves are then described using a symbolic representation based on their winding around the two remaining binary collision points. As shown in Figure 3, clockwise and counterclockwise rotations around the right and left collision points are denoted by letters “a” and “b”, respectively, while their inverses are represented by “A” and “B”. The sequence of these letters forms a “free group element (word)”, which uniquely describes the orbit’s topological structure [24, 44]. Using this topological classification, the free group word of the BHH family of periodic orbits is “a”, while the free group word for the figure-eight solution is “abAB” [24]. For more details on the topological identification method, please refer to ref. [44].

An alternative simpler symbolic representation can be constructed by labeling each syzygy with the body that occupies the middle position during the collinear configuration. The resulting sequence of numbers 1, 2, and 3 is referred to as the “syzygy sequence” and provides another way for characterizing the topological properties of the orbit [46]. For the figure-eight solution, its free group word is “abAB”, while its syzygy sequence [25] is 123123.

4 Analytical periodic orbits

Euler [8] identified a class of periodic solutions in which

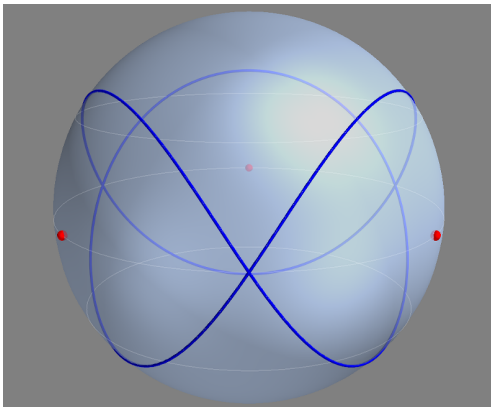


Figure 1 (Color online) The shape-space sphere. Solid curve: the figure-eight periodic orbit; red points: the two-body collisions.

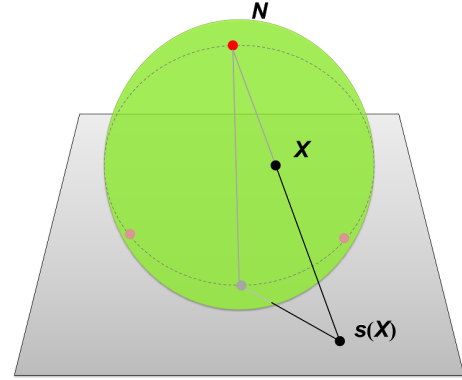


Figure 2 (Color online) The stereographic projection of the shape sphere onto a plan, the red points denote the two-body collisions.

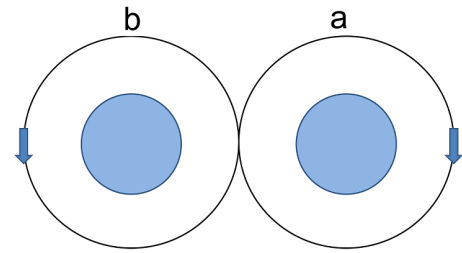


Figure 3 (Color online) The two words (a, b) of the free group elements.

three celestial bodies remain collinear while moving elliptically around their common center of mass, as shown in Figure 4(a). However, these collinear periodic solutions are unstable, meaning even small perturbations can disrupt them. In terms of topological classification, the Euler’s solution is always at the syzygy moment. Lagrange [9] discovered a different type of periodic solutions where the three bodies form an equilateral triangle that rotates as they move along elliptical orbits, as shown in Figure 4(b). Remarkably, these equilateral triangle solutions exist for any combination of mass. A well-known example of Lagrange’s solution is the asteroid Achilles, which forms an equilateral triangle with Jupiter and the Sun in its orbit.

Broucke and Lass [47] introduced relative coordinates to describe the motion of the three-body problem. By defining the relative position vectors as $\mathbf{s}_1 = \mathbf{r}_3 - \mathbf{r}_2$, $\mathbf{s}_2 = \mathbf{r}_1 - \mathbf{r}_3$, $\mathbf{s}_3 = \mathbf{r}_2 - \mathbf{r}_1$ and using the identity $\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3 = 0$, the equations of motion (eq. (1)) are transformed into

$$\ddot{\mathbf{s}}_i = -GM \frac{\mathbf{s}_i}{s_i^3} + m_i \mathbf{Q}, \quad i = 1, 2, 3, \quad (5)$$

where $M = m_1 + m_2 + m_3$ is the total mass, and the vector $\mathbf{Q}$ is given by

$$\mathbf{Q} = \sum_{i=1}^3 \frac{\mathbf{s}_i}{s_i^3}. \quad (6)$$

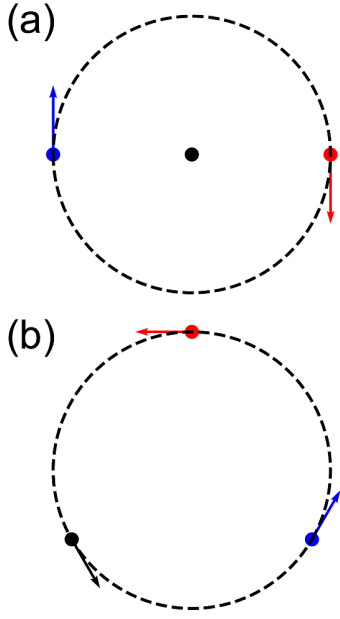


Figure 4 (Color online) The analytical solutions of the three-body problem: (a) Euler's solution; (b) Lagrange's solution.

In the Euler's solution, three bodies are collinear at all times. Let m_2 lie between the other two bodies. This configuration allows the relative position vectors to be expressed as

$$s_1 = \alpha s_3, \quad s_2 = -(1 + \alpha)s_3, \quad (7)$$

where α is a positive scalar to be determined. Substituting eq. (7) into eq. (5), a fifth-degree polynomial for α is obtained [48]:

$$(m_1 + m_2)\alpha^5 + (3m_1 + 2m_2)\alpha^4 + (3m_1 + m_2)\alpha^3 - (m_2 + 3m_3)\alpha^2 - (2m_2 + 3m_3)\alpha - (m_2 + m_3) = 0. \quad (8)$$

The scalar α can be determined by solving this equation. Using this value, the equation of motion for relative position vector s_3 is

$$\ddot{s}_3 = -\frac{m_2 + m_3(1 + \alpha)^{-2}}{m_2 + m_3(1 + \alpha)} \frac{GMs_3}{s_3^3}. \quad (9)$$

This represents the equation of motion for a two-body problem. As a result, the three bodies move elliptically, with the same eccentricity and period, around their common center of mass. In the Lagrange's solution, the relative distances between any two bodies are equal, i.e., $s_{12} = s_{13} = s_{23}$. This symmetry causes the vector $\mathbf{G}$ to vanish ($\mathbf{G} = 0$), and eq. (5) decouple into a set of identical two-body equations:

$$\ddot{s}_i = -GMs_i/s_i^3, \quad i = 1, 2, 3. \quad (10)$$

Therefore, each of the three bodies follows an elliptical orbit around the common center of mass, forming an equilateral triangle configuration at all times, as shown in Figure 4(b).

5 Numerical methods for the three-body problem

5.1 Numerical integrators for the three-body problem

Traditional numerical methods, such as the Runge-Kutta-Fehlberg algorithm [49] and the Bulirsch-Stoer method [50] with adaptive time stepsize are widely used to integrate the differential equations of the three-body problem [26, 44, 51]. However, the three-body problem is inherently chaotic, meaning that its solutions are highly sensitive to initial conditions and numerical errors. To address this challenge, Liao [28] proposed the CNS [52], which combines high-order Taylor series methods [53–55] with multi-precision computation [56] to reduce numerical error and ensure the convergence of numerical simulations of chaotic systems. CNS has been employed to obtain reliable numerical results for the Lorenz equation [28], the general three-body problem [29], spatio-temporal chaos [57] and turbulences [58, 59].

The differential equations of the three-body problem can be formulated as follows:

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X}), \quad \mathbf{X}(0) = \mathbf{X}_0, \quad (11)$$

where t denotes time, $\mathbf{X}(t)$ represents the state vector at time t with initial condition $\mathbf{X}_0$, and $\mathbf{f}$ is a nonlinear function. The solution is computed numerically using a Taylor series method. For a given time step h , the solution at time $t_{n+1} = t_n + h$ is approximated by

$$\mathbf{X}(t_n + h) \approx \mathbf{X}(t_n) + \sum_{n=1}^M \frac{\mathbf{X}^{(n)}(t_n)}{n!} h^n, \quad (12)$$

where $\mathbf{X}^{(n)}(t_n)$ denotes the n -th derivative of $\mathbf{X}(t)$ evaluated at t_n . The truncation error in this approximation can be controlled by increasing the order M of the Taylor series expansion and/or by reducing the time step h .

A critical aspect of CNS is controlling rounding errors, which arise from the finite precision of computer arithmetic. Early work laid the groundwork for understanding these errors [60, 61], while Henrici [62] highlighted their amplified impact in solving differential equations compared to algebraic calculations. The IEEE 754 standard (1985, revised 2008 [63]) introduced standardized rounding methods for floating-point operations, including quadruple precision (128-bit). However, even this level of precision is often insufficient for reliable numerical simulations of chaotic systems like the three-body problem. Therefore, CNS employs arbitrary precision arithmetic libraries like the GNU multiple

precision arithmetic library (GMP) [64] and multiple precision floating-point reliable library (MPFR) [65]. These open-source libraries enable calculations with user-defined precision, effectively controlling rounding errors and ensuring the reliability of numerical results.

The CNS achieves high numerical accuracy by combining three key strategies: (1) reducing truncation error through high-order Taylor series expansions and small time steps; (2) controlling rounding error using arbitrary-precision arithmetic, implemented via GMP and MPFR libraries; and (3) verifying convergence and reliability by comparing results obtained with progressively smaller error tolerances. While the CNS ensures reliable results with high accuracy over long periods of time, it requires significantly more computational resources. The computational cost of the CNS method is roughly more than 100 times greater than traditional methods with double precision, depending on the required precision and integration time of the solutions. A computer cluster or supercomputer is typically necessary for conducting extensive searches for periodic solutions of the three-body problem [27].

Due to the inherent chaos of the three-body problem, reliable computation of trajectories is a critical step in the search for periodic orbits. Traditional methods using double precision typically have a tolerance limit of approximately 10^{-16} , which is comparable to the round-off error dictated by the 64-bit storage format for double precision data. However, this level of tolerance is often insufficient for the highly sensitive nature of the three-body problem. The CNS method addresses this limitation by employing high-order Taylor series expansions and multiple-precision arithmetic, allowing for an adjustable tolerance tailored to the specific problem at hand. Li and Liao [27] demonstrated that the CNS method identified 243 more periodic orbits of the three-body problem than the widely used ODE solver *dop853* [49]. This finding highlights the significant impact of numerical noise, which can result in the loss of many periodic orbits when using traditional double precision methods. Furthermore, a numerical comparison was made between the CNS and traditional double precision methods for chaotic nonlinear equations [66].

5.2 Numerical searching method for periodic orbits of the three-body problem

The high dimensionality of the three-body problem makes it challenging to search for periodic orbits in the full phase space. To address this, initial configurations such as collinear or free-fall setups have been employed to reduce the search space. By reducing the search space to two dimensions, grid search methods [44, 67] become effective tools for identifying approximate initial conditions of periodic orbits. In some

cases, certain initial conditions can be constrained to specific values based on the initial configuration. For example, Šuvakov and Dmitrašinović [44] adopted a collinear symmetric initial configuration with parallel velocities. In this setup, the initial positions and velocities of the three bodies are given as follows: $\mathbf{r}_1(0) = -\mathbf{r}_2(0) = (1, 0)$, $\mathbf{r}_3(0) = (0, 0)$, $\dot{\mathbf{r}}_1(0) = \dot{\mathbf{r}}_2(0) = (v_x, v_y)$, $\dot{\mathbf{r}}_3(0) = -2\dot{\mathbf{r}}_1(0)$. Here, the initial conditions are controlled by two unknown parameters v_x and v_y . To identify approximate initial conditions, a grid search can be performed over the discretized v_x - v_y plane. The approximate initial conditions are considered valid when the return proximity function (eq. (3)) falls below a prescribed tolerance. More initial configurations will be discussed in Section 6.

These approximate initial conditions are further refined for better accuracy using optimization techniques, such as the gradient descent algorithm [44], the single shooting method [27] based on the Newton-Raphson method [68] which is integrated with the CNS method [29]. Due to the inherent chaos of the three-body problem, numerical noise has a significant impact on the accuracy of solutions. Consequently, it is essential to strictly control numerical noise during the solution process. The CNS addresses this issue by employing high-order Taylor series methods to ensure that truncation errors remain below a desired tolerance (e.g., 10^{-90}). Additionally, the CNS utilizes multiple-precision arithmetic (e.g., 120 digits) to minimize round-off errors, thereby reducing the influence of numerical noise that is unavoidable in traditional double-precision methods. To further mitigate the effects of chaos, the CNS validates results by comparing simulations performed under different numerical noise levels. Using this approach, Li and Liao [27] obtained periodic orbits with differential correction tolerances in Newton iterations less than 10^{-6} . This tolerance of 10^{-6} in the differential correction is chosen as a practical balance between computational efficiency and the accuracy required to identify periodic solutions. It can also be adjusted to other values depending on the precision required for the specific applications or analyses.

Additionally, numerical continuation methods [69] have been applied to extend periodic orbits across varying parameters, such as mass ratios [32]. Exploiting the symmetry properties of periodic solutions further enhances search efficiency by reducing the period to half [36, 70].

6 Numerical periodic three-body orbits

Numerical searching for periodic orbits is challenging due to the high dimensionality of the three-body problem. The motion equation of the planar three-body problem contains 12 first-order differential equations. Conserved quantities such

as the center of mass, angular momentum, and energy reduce the effective degrees of freedom to six. However, searching periodic orbits in this reduced space is even computationally demanding. Researchers often restricted their search to lower-dimensional subspaces defined by specific initial configurations, such as collinear, isosceles collinear, or free-fall setups, reducing the search space to two dimensions.

6.1 Collinear initial configuration

In the collinear initial configuration, the three bodies start aligned along a straight line, with initial positions $\mathbf{r}_1(0) = (x_1, 0)$, $\mathbf{r}_2(0) = (x_2, 0)$, and $\mathbf{r}_3(0) = (x_3, 0)$, and initial velocities perpendicular to the line: $\dot{\mathbf{r}}_1(0) = (0, v_1)$, $\dot{\mathbf{r}}_2(0) = (0, v_2)$, and $\dot{\mathbf{r}}_3(0) = (0, v_3)$.

In the 1970s, researchers discovered a new family of periodic solutions for this configuration [18–20]. These periodic orbits, categorized by their topological properties, are known as the Broucke-Hadjidemetriou-Hénon (BHH) family [24]. Broucke [18] introduced two reductions for the planar three-body problem and used heliocentric coordinates to obtain relative periodic solutions. Around the same time, Hénon [20] identified a one-parameter family of periodic orbits in the equal-mass planar three-body problem. These orbits originated from the rectilinear orbits discovered by Schubart [71] within a rotating reference frame. Trajectories of the BHH family are shown in Figure 5. The existence of the BHH orbit was proven by using variational methods [72]. Kuang and Yan [73] investigated the three-body problem with two equal masses involving the collinear configurations and isosceles configurations and proved the existence of these periodic orbits via variational method.

Further advancements came from Janković and Dmi-trašinović [26], who presented 57 topological satellites of the BHH family for equal-mass three-body problem with non-zero angular momentum. Janković et al. [33] detailed numerical methods for identifying these periodic orbits. More recently, Liao et al. [74] presented a road map to effectively obtain periodic orbits of the three-body system with varying masses using an ANN model. By training an ANN model on a set of periodic orbits within a finite mass range, they demonstrated the ANN model can predict periodic orbits for a wider range of masses. Figure 6 illustrates the dynamical trajectories of these satellites of BHH periodic orbits with various masses.

Montgomery [75] proved that the three bodies have collinear instant for the three-body problem with zero angular momentum, except in the case of Lagrange's solution. This makes the collinear initial configuration a natural and widely accepted choice. As described above, the initial velocities are set perpendicular to the line defined by initial

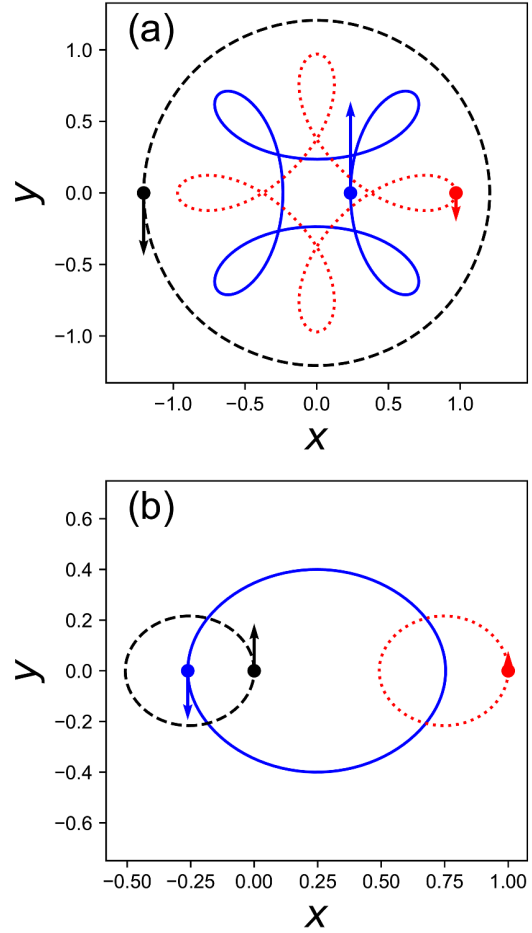


Figure 5 (Color online) Broucke-Hadjidemetriou-Hénon (BHH) family solution: (a) Broucke's solution in fixed reference frame [18]; (b) Hénon's solution in the rotating reference frame [20].

positions of the three bodies. These periodic orbits belong to the BHH family and their satellites orbits. Consequently, the study of periodic solutions to the three-body problem becomes even more significant when no velocity constraints are imposed, as this can reveal a broader range of solutions and dynamic behaviors.

6.2 Eulerian initial configuration

In the Eulerian initial configuration, the three bodies are aligned with one body positioned exactly midpoint between the other two. This configuration has led to lots of important discoveries. In 1993, Moore [21] discovered the famous figure-eight periodic solution with this configuration. This solution was later rediscovered and rigorously proved by Chenciner and Montgomery [22]. As shown in Figure 7, the three bodies of the figure-eight solution follow the same path. Subsequent studies extended this solution to rotating reference frames [76–79]. Chen [80] applied the variational approach to prove the existence of retrograde orbits for the

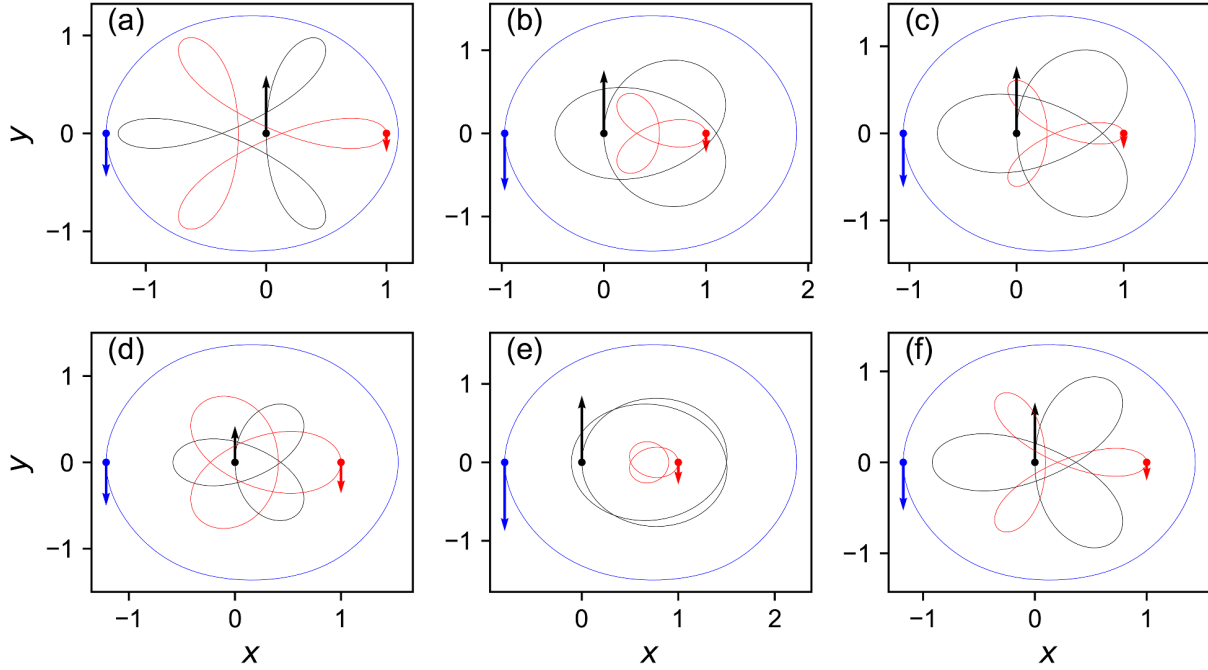


Figure 6 (Color online) The satellite orbits of the BHH families in the three-body problem with various masses m_1 , m_2 , and $m_3 = 1$ in a rotating frame of reference obtained by using an artificial neural network (ANN) model [74]: (a) $m_1 = 1.0124$ and $m_2 = 0.9968$; (b) $m_1 = 0.5312$ and $m_2 = 2.2837$; (c) $m_1 = 0.8056$ and $m_2 = 2.0394$; (d) $m_1 = 0.3916$ and $m_2 = 0.8341$; (e) $m_1 = 0.1472$ and $m_2 = 3.4219$; (f) $m_1 = 0.8413$ and $m_2 = 1.4155$.

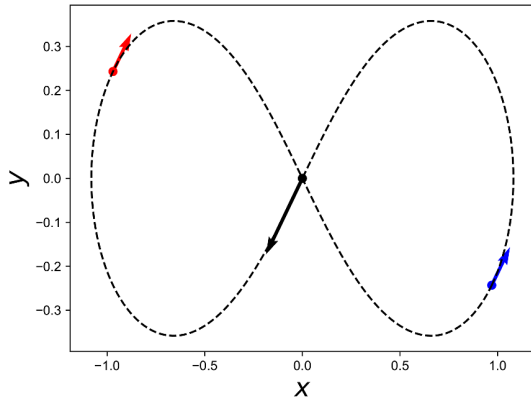


Figure 7 (Color online) Figure-eight solution. The three bodies follow the same path.

three-body problem with various choices of masses, including retrograde triple stars, binary stars with one outer planet, and binary stars with one planet orbiting around one primary mass. Yu [39] completed the proof for the existence of the figure-8 solution in the three-body problem with two equal masses.

In 2013, Šuvakov and Dmitrašinović [24] conducted a numerical search for periodic orbits in the planar equal-mass three-body problem with zero angular momentum. Their initial configuration is given by: $\mathbf{r}_1(0) = (-1, 0)$, $\mathbf{r}_2(0) = (1, 0)$, $\mathbf{r}_3(0) = (0, 0)$, $\dot{\mathbf{r}}_1(0) = \dot{\mathbf{r}}_2(0) = (v_1, v_2)$, and $\dot{\mathbf{r}}_3(0) = -2\dot{\mathbf{r}}_1(0)$.

The initial condition of three bodies is determined by two parameters v_1 and v_2 , thus, the searching space is reduced to two dimensions. By refining initial conditions using a gradient descent method, they identified 13 distinct periodic orbits, classified into four topological groups, three of which were newly discovered. Some orbits are shown in Figure 8. Please visit the website ¹⁾ for more details.

Their search methods and topological classification were detailed in subsequent work [44], where they introduced relative Jacobian three-body coordinates and the shape sphere to reduce the dimensionality of the problem. Building on these findings, Li and Liao [81] investigated the stability of these newly discovered periodic orbits using CNS, finding that seven of the orbits are unstable. Hudomal [82] expanded the known families of periodic solutions by reporting 25 families, including the 11 previously discovered by Šuvakov and Dmitrašinović [24]. Iasko and Orlov [51] performed numerical investigation and found near-periodic orbits in a region of initial conditions for the equal-mass three-body system with zero angular momentum.

Šuvakov and Shibayama [38] introduced a numerical method specifically designed to find choreography periodic orbits, where all three bodies follow the same trajectory. Their work identified 20 distinct choreographies, three of which were not satellites of the figure-eight solution, as

¹⁾ <http://three-body.ipb.ac.rs/>.

shown in Figure 9. Rose [37] reported 90 periodic collisionless orbits in the same initial configuration, which includes the figure-eight family and other families reported in refs. [38, 44, 82]. In 2017, Li and Liao [27] applied CNS to discover 695 families of periodic equal-mass three-body orbits, including the figure-eight and 11 orbits found by Šuvakov and Dmitrašinović [24], along with over 600 new ones. They employed a grid search method to obtain the approximated initial conditions and refine them with the Newton-Raphson method. Their results indicate that smaller grid sizes and longer periods could reveal even more periodic solutions, suggesting the potential for further discoveries. Some of these complex periodic orbits are shown in Figure 10. Li

et al. [30] later extended their numerical search to the system with two equal-mass bodies, identifying over one thousand new families of periodic orbits. They also used numerical continuation methods to generalize these orbits to cases where all three bodies have different masses. For more details, please visit the website ²⁾.

Periodic orbits were studied numerically in other potential systems as well. Dmitrašinović et al. [40] explored the periodic orbits of the three-body problem with a Jacobi-Poincaré ‘strong’ potential $\sum_{i>j} -1/r_{ij}^2$, discovering 24 topological distinct orbits. Šindik et al. [83] identified approximately 100 distinct collisionless periodic three-body orbits with zero angular momentum in the Coulomb potential in vacuo, involv-

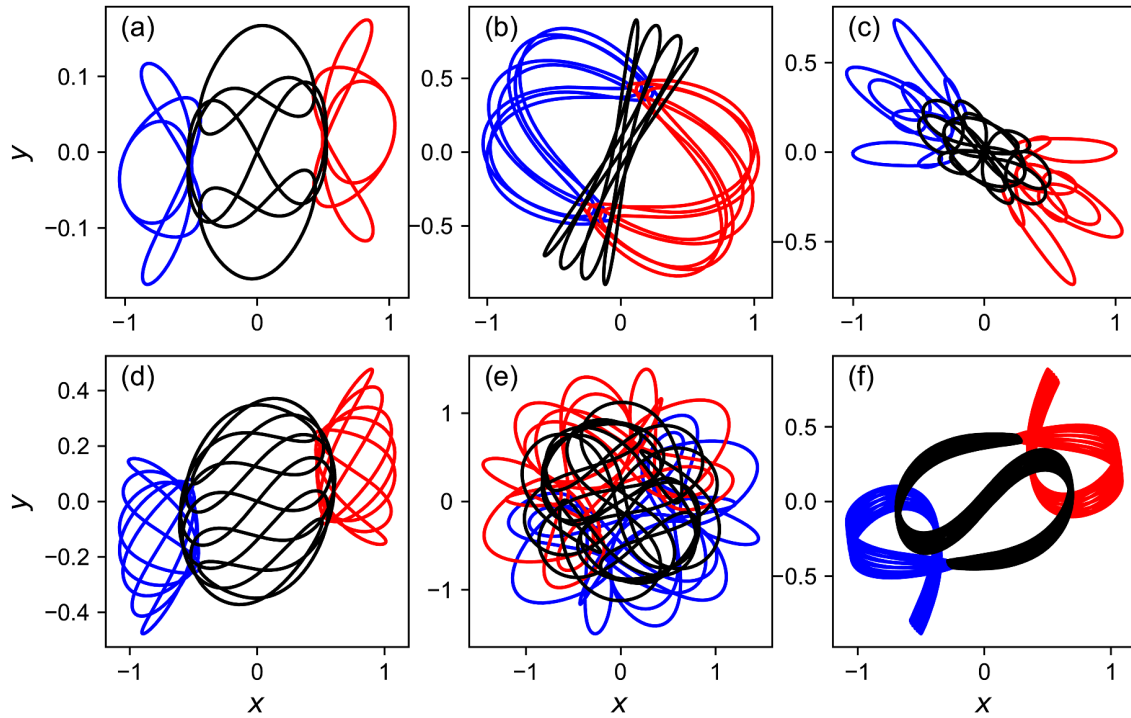


Figure 8 (Color online) Some of periodic orbits of the equal-mass three-body problem discovered by Šuvakov and Dmitrašinović [24]: (a) I.A.1 butterfly I; (b) I.B.7 dragonfly; (c) I.B.5 goggles; (d) I.B.3 butterfly III; (e) II.B.1 yarn; (f) II.C.3a yin-yang II.

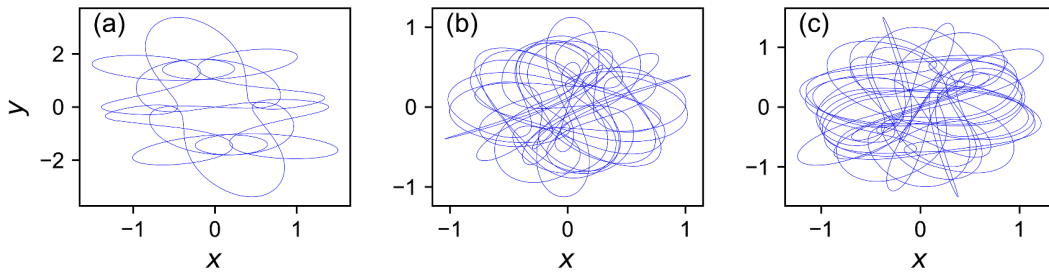


Figure 9 (Color online) Three choreography periodic orbits with non-slalom homotopy classes [38]: (a) No. 22; (b) No. 23; (c) No. 24.

²⁾ <https://github.com/sjtu-liao/three-body>.

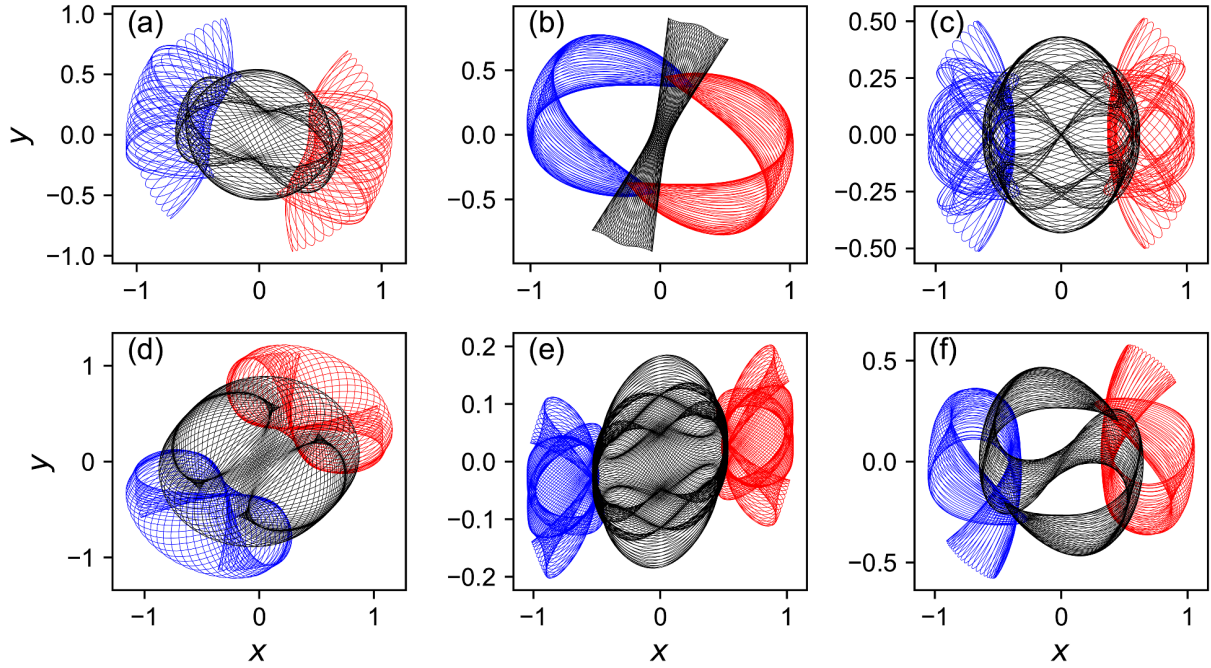


Figure 10 (Color online) Some of complex periodic three-body orbits with equal masses [27]: (a) $I.A_{77}^{i.c.}$; (b) $I.A_{100}^{i.c.}$; (c) $I.A_{115}^{i.c.}$; (d) $I.B_{102}^{i.c.}$; (e) $I.B_{155}^{i.c.}$; (f) $I.C_{156}^{i.c.}$.

ing equal-mass ions with identical absolute charge values. The classical periodic orbits of three bodies in Efimov potential were also investigated [84].

In the Eulerian initial configuration, an increasing number of periodic three-body orbits have been discovered with complex topological structures. These findings reflect significant advancements in understanding periodic orbits in the three-body problem with advanced numerical methods. Furthermore, these discoveries suggest that even more periodic orbits could be identified with finer grid searches and longer orbital periods. In current studies, the initial velocities of the three bodies are typically constrained to have the same direction. Therefore, it is worthwhile to investigate the three-body problem in the Eulerian initial configuration when the velocities of the three bodies are allowed to have different directions, as this may reveal new families of periodic orbits.

6.3 Free-fall initial configuration

The free-fall three-body problem, where three bodies start with zero initial velocities, was first numerically studied by Burrau [12] in 1913. These results showed one body escaping while the other two formed a binary system. The first found periodic solution for this configuration involves a binary collision [16]. Standish [17] later discovered the first non-colliding periodic solution in this configuration. The mathematical proof for the existence of periodic free-fall three-body orbits with binary collision was provided in the

isosceles initial configuration [85]. Chen [86] later employed various shooting arguments to find six types of periodic orbits in the isosceles three-body system with zero initial velocities. To reduce the dimensionality of the searching space, a numerical search for free-fall orbits was conducted in the Hill surface [87]. Tanikawa and Mikkola [88] discovered periodic free-fall three-body orbits involving binary collisions with initial conditions on the circular boundary of Agekyan and Anosova's region [35]. Yasko and Orlov [34] searched for near-periodic free-fall equal-mass three-body orbits in the Agekyan-Anosova's region and obtained 22 periodic orbits. Further investigation was conducted to hunt for periodic orbits on the Hill surface in the three-body system with equal masses [89]. In more recent work, Li and Liao [31] applied CNS and the Newton-Raphson method to identify 316 collisionless periodic orbits for the free-fall three-body problem with various masses in the Agekyan-Anosova's region. Building on these advances, Hristov et al. [36] improved the numerical method proposed by Li and Liao [27] to obtain 12409 distinct periodic solutions with a finer search grid and some technical improvements for the equal-mass free-fall three-body problem.

Thousands of periodic orbits of the free-fall three-body problem have been discovered by using sophisticated numerical methods. However, all of these periodic orbits are inherently unstable. In the free-fall three-body problem, the trajectories are highly unstable and typically evolve into a disruption where a binary system forms while the third body

escapes. It is particularly intriguing to study the distribution of disruptions in the vicinity of periodic free-fall orbits, as this could provide deeper insights into the dynamics of such systems.

7 Linear stability of periodic orbits

The stability of periodic orbits is essential in understanding the long-term dynamical behavior of the three-body system. Stability determines whether a particular orbit persists when subjected to small perturbations. One common method for assessing this is linear stability analysis, which examines the eigenvalues of the monodromy matrix. If all eigenvalues lie on the unit circle in the complex plane, the orbit is considered linearly stable.

In the planar three-body problem, the system's state can be described by a 12-dimensional vector $X(t)$. Let $X_{PO}(t)$ represent a periodic solution with period T . If a small perturbation $\delta X(0)$ is added to the initial condition, the new initial state becomes $X(0) = X_{PO}(0) + \delta X(0)$. After one period, the system evolves to $X(T) = X_{PO}(T) + \delta X(T)$. The relationship between the initial perturbation $\delta X(0)$ and the deviation $\delta X(T)$ after one period is expressed as

$$\delta X(T) = \Phi(T)\delta X(0), \quad (13)$$

where $\Phi(T)$ is the monodromy matrix. The matrix Φ can be computed by integrating the system's linearized equations along the periodic orbit:

$$\frac{d\Phi}{dt} = \frac{\partial f}{\partial X} \Big|_{X_{PO}} \Phi, \quad \Phi(0) = I. \quad (14)$$

The eigenvalues λ of the monodromy matrix $\Phi(T)$ determine the linear stability of the orbit [90]. These eigenvalues typically appear in pairs or groups of four ($\lambda, \lambda^{-1}, \lambda^*, \lambda^{*-1}$) and can be classified into several types:

- (1) Elliptical: $\lambda = \exp(\pm 2\pi i \nu)$, where the winding number ν is a real and eigenvalues lie on the unit circle.
- (2) Marginally stable: $\lambda = +1$ or $\lambda = -1$.
- (3) Hyperbolic: $\lambda = \exp(\pm \mu)$ or $\lambda = -\exp(\pm \mu)$, where $\mu > 0$ is the Lyapunov exponent of the periodic orbit.
- (4) Loxodromic type: $\lambda = \exp(\pm \mu \pm i\nu)$ where μ and ν are real numbers.

Simó [91] demonstrated that the well-known figure-eight periodic solution exhibits elliptical stability, with eigenvalues $\lambda_j = \exp(2\pi i \nu_j)$, where $\nu_1 = 0.00842272$ and $\nu_2 = 0.29809253$.

The dimensionality of the planar three-body problem can be reduced to eight due to the fixed center of mass. Képele and Simó [92] developed a theory to assess the linear stability of periodic orbits in the three-body problem using the monodromy matrix. Given the monodromy matrix

$A = \Phi(T)$, the following characteristic equation is

$$Y^2 - (\alpha - 4)Y + \beta - 4\alpha + 8 = 0, \quad (15)$$

where $\alpha = \text{trace}(A) = \sum_{i=1}^8 a_{ii}$ and $\beta = \sum_{1 \leq i < j \leq 8} (a_{ii}a_{jj} - a_{ij}a_{ji})$, with a_{ij} representing elements of the monodromy matrix A . Let Y_1 and Y_2 be the solutions of the above equation. If $\Delta = (\alpha - 4)^2 - 4(\beta - 4\alpha + 8) > 0$, $|Y_1| < 2$ and $|Y_2| < 2$, then all eigenvalues of the monodromy matrix A lie on the unit circle [92].

Hénon [20] established the linear stability of the Broucke-Hadjidemetriou-Hénon family of periodic orbits with non-vanishing angular momentum. Dmitrašinović et al. [40] used Floquet exponents to study linear stability of newly discovered periodic three-body orbits, finding that 21 of them are linearly stable. Li et al. [30] numerically obtained 1349 families of periodic orbits in systems with two equal masses, of which 23 families are linearly stable and 545 ones are marginally stable. Vasiljević et al. [41] conducted a comprehensive analysis of publicly available planar periodic three-body orbits with zero angular momentum. They identified compact “island” of marginally stable and stable orbits within the parameter space, providing boundaries in two additional dimensions.

Stability analysis is particularly important when studying systems with varying mass ratios. Galán et al. [93] used numerical continuation methods and characteristic multipliers to investigate the stability properties of the figure-eight solution, finding it stable within a very narrow mass interval (approximately 10^{-5}). More recently, Li et al. [32] employed a numerical continuation method to discover a family of periodic non-hierarchical three-body orbits with unequal masses, revealing a larger stable region compared to the figure-eight orbits. This suggests that stable non-hierarchical orbits have significant potential for future observation.

In summary, the stability of periodic orbits in the three-body problem is crucial for understanding their long-term dynamics. Through linear stability analysis and numerical methods, researchers have identified both stable and marginally stable solutions across a wide range of configurations, including systems with equal and unequal masses.

8 Kepler's third law for the periodic three-body orbits

Kepler formulated three laws of planetary motion, which describe the behavior of planets in the solar system. His third law particularly states that for planets orbiting the Sun in elliptical paths, the square of the orbital period T is proportional to the cube of the semi-major axis a of the orbit, expressed as $T^2 \propto a^3$. While Kepler's laws apply perfectly to

the two-body problem, where bodies move in elliptical orbits. For the three-body problem, only the Lagrange periodic solutions exhibit motion resembling elliptical orbits. Most other periodic solutions do not follow such paths. As a result, the concept of a semi-major axis lacks a clear definition for the three-body problem.

Dmitrašinović and Šuvakov [25] tackled this issue by proposing a way to define the semi-major axis a in the three-body system using its total energy. They introduced a size scale of the system by averaging the hyper-radius R over one period. The semi-major axis a of the three-body system is then defined as the mean hyper-radius $\bar{R}$, which satisfies the relation: $\frac{1}{\bar{R}} = \frac{1}{T} \int_0^T \frac{dt}{R(t)}$. According to the virial theorem [94], the mean size $\bar{R}$ is proportional to inverse of the system's total energy $|E|^{-1}$, i.e., $\bar{R} \propto |E|^{-1}$. Consequently, Kepler's third law can be extended to the three-body system in the form $T \propto |E|^{-3/2}$, or equivalently $T|E|^{3/2} = \text{const}$, where the constant depends on both the family of the three-body orbit and the system's angular momentum. Janković and Dmitrašinović [26] investigated the relationship between angular momentum and topological properties of the BHH family of periodic orbits and found that the $L(T/k)$ curve of their satellite periodic orbits is nearly indistinguishable from the $L(T)$ curve for the non-satellite solutions, as shown in Figure 11.

Li and Liao [27] further found that the 695 families of equal-mass three-body systems approximately conformed to this generalized version of Kepler's third law. Specifically, they observed that the average period $\bar{T} = T/L_f$ with L_f denotes the length of the free group element of the three-body periodic orbit, satisfies the relation $\bar{T}|E|^{3/2} = \text{const}$. For these families, the scale-invariant average period $\bar{T}^* = \bar{T}|E|^{3/2}$ remains roughly constant across the different orbits. Further investigation explored periodic solutions involving two equal-

mass bodies and a third body with varying mass [30]. It was discovered that the scale-invariant average period $\bar{T}^* = \bar{T}|E|^{3/2}$ changes linearly with the mass of the third body. This result prompted Sun [95] to derive a generalized version of Kepler's third law for the n -body problem using dimensional analysis. Semay and Willemyns [96] also found quasi Kepler's third law for quantum many-body systems.

These findings demonstrate the Kepler's third law in n -body systems, offering valuable insights into the behavior of periodic orbits in the three-body problem. This extension of classical mechanics provides a deeper understanding of how energy and period are linked in complex gravitational interactions. As these periodic orbits were identified within a limited period, the extension of Kepler's third law suggests that more periodic orbits may be discovered by exploring longer periods. Although this can not directly predict the initial conditions of new periodic orbits, the period corresponding to any given total energy can still be determined. For astrophysical observations, it could provide a potential means to estimate the masses of bodies within periodic three-body system.

9 Discussion and conclusion

In recent decades, the discovery of thousands of periodic solutions to the three-body problem, spanning both equal and unequal masses, with or without angular momentum, has significantly deepened our understanding of the three-body problem. Several open questions and promising areas for future research remain:

(1) The development of extensive databases of periodic orbits presents an exciting avenue for further exploration. A key question is whether machine learning techniques can be leveraged to predict new periodic solutions [52]. Machine learning methods have emerged as powerful tools for tackling the computational challenges of the three-body problem. Breen et al. [97] demonstrated that deep neural networks can predict chaotic three-body trajectories with accuracy comparable to numerical integration. Similarly, Yang et al. [98] utilized ANNs to accelerate the computation of the Jovian-moon flyby map. Therefore, by analyzing the patterns and characteristics from the known periodic orbits, machine learning algorithms may offer a novel approach to identify new families of periodic solutions.

(2) Most research has concentrated on planar three-body systems, but the application of numerical continuation methods to three-dimensional periodic orbits could reveal richer, more complex dynamics. Investigating the dynamical properties of such orbits in the general three-body problem presents a compelling direction for future study, with the

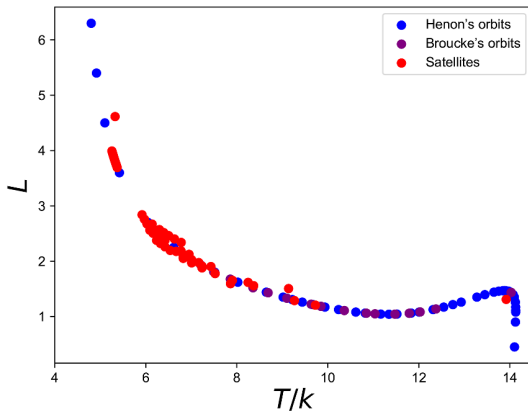


Figure 11 (Color online) The angular momentum L versus T/k at fixed energy $E = -0.5$ for the Broucke-Hadjidemetriou-Hénon's family of periodic orbits, where T and k denote the period and the topological exponent k , respectively. The $L(T/k)$ curve of satellite orbits approximately coincides with the BHH periodic orbits' $L(T)$ curve [26].

potential to uncover new behaviors not observed in two-dimensional systems.

(3) The KAM (Kolmogorov-Arnold-Moser) theory may provide critical insights into the long-term stability of periodic orbits under small perturbations. For the well-known figure-eight solution, KAM stability was investigated by Simó [91]. Kapela and Simó [99] later proved the KAM stability of the figure-eight orbit by means of a computer assisted method. Although the linear stability of many newly discovered periodic orbits has been extensively studied, their nonlinear stability remains largely unexplored. A deeper investigation into the nonlinear stability of these newly found orbits is essential for a more comprehensive understanding of their dynamical behavior.

(4) Observed three-body systems in the universe are typically hierarchical, with one body orbiting a pair of closer bodies. However, recent discoveries of stable non-hierarchical solutions, particularly in systems with large mass ratios, raise the intriguing possibility that such orbits may exist in nature. The potential presence of non-hierarchical periodic three-body systems opens up new avenues for both observational astronomy and theoretical investigation.

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